BLM2041 Signals and Systems

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Digital Signal Processing

FIR Filtering and Frequency Response

LECTURE OBJECTIVES

- INTRODUCE FILTERING IDEA
  - Weighted Average
  - Running Average
- FINITE IMPULSE RESPONSE FILTERS
  - **FIR** Filters
  - Show how to **compute** the output $y[n]$ from the input signal, $x[n]$

LECTURE OBJECTIVES

- SINUSOIDAL INPUT SIGNAL
  - DETERMINE the FIR FILTER OUTPUT
- FREQUENCY RESPONSE of FIR
  - PLOTTING vs. Frequency
  - MAGNITUDE vs. Freq
  - PHASE vs. Freq

MAG

PHASE

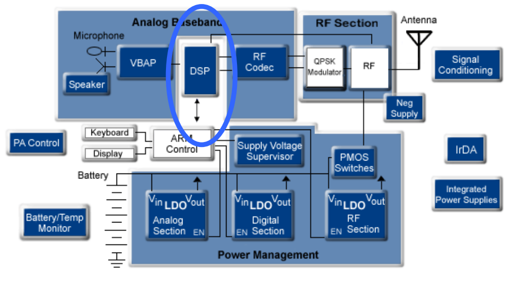
$$H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})| e^{j\angle H(e^{j\hat{\omega}})}$$

DIGITAL FILTERING

$x(t) \rightarrow$ **A-to-D** $\xrightarrow{x[n]}$ **COMPUTER** $\xrightarrow{y[n]}$ **D-to-A** $\rightarrow y(t)$

- CONCENTRATE on the COMPUTER
  - PROCESSING ALGORITHMS
  - SOFTWARE (MATLAB)
  - HARDWARE: DSP chips, VLSI
- DSP: DIGITAL SIGNAL PROCESSING

Digital Cell Phone



Now it plays video

DISCRETE-TIME SYSTEM

$x[n]$ → **COMPUTER** → $y[n]$

- OPERATE on $x[n]$ to get $y[n]$
- WANT a **GENERAL** CLASS of SYSTEMS
  - **ANALYZE** the SYSTEM
    - TOOLS: TIME-DOMAIN & FREQUENCY-DOMAIN
  - **SYNTHESIZE** the SYSTEM

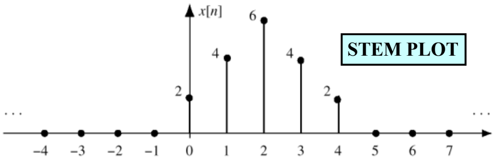
D-T SYSTEM EXAMPLES

$x[n]$ → **SYSTEM** → $y[n]$

- EXAMPLES:
  - POINTWISE OPERATORS
    - SQUARING: $y[n] = (x[n])^2$
  - RUNNING AVERAGE
    - RULE: "the output at time n is the average of three consecutive input values"

DISCRETE-TIME SIGNAL

- $x[n]$ is a LIST of NUMBERS
  - INDEXED by "n"



3-PT AVERAGE SYSTEM

- ADD 3 CONSECUTIVE NUMBERS
  - Do this for each "n"

the following input–output equation **Make a TABLE**

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0
$y[n]$	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

n=0 $y[0] = \frac{1}{3}(x[0] + x[1] + x[2])$

n=1 $y[1] = \frac{1}{3}(x[1] + x[2] + x[3])$

3-PT AVERAGE SYSTEM

INPUT SIGNAL

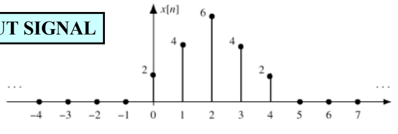


Figure 5.2 Finite-length input signal, $x[n]$.

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

OUTPUT SIGNAL

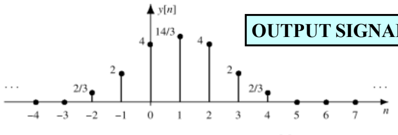


Figure 5.3 Output of running average, $y[n]$.

PAST, PRESENT, FUTURE

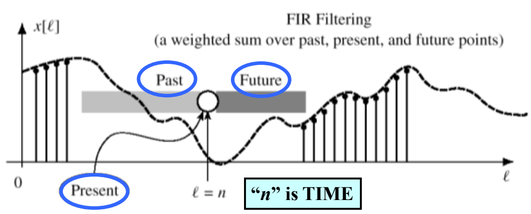


Figure 5.4 The running-average filter calculation at time index n uses values within a sliding window (shaded). Dark shading indicates the future ($\ell > n$); light shading, the past ($\ell < n$).

ANOTHER 3-pt AVERAGER

- Uses "PAST" VALUES of $x[n]$
  - IMPORTANT IF "n" represents REAL TIME
    - WHEN $x[n]$ & $y[n]$ ARE STREAMS

$$y[n] = \tfrac{1}{3}(x[n] + x[n-1] + x[n-2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	6	7	$n > 7$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0	0	0
$y[n]$	0	0	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - For example, $b_k = \{3, -1, 2, 1\}$

$$y[n] = \sum_{k=0}^{3} b_k x[n-k]$$
$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$

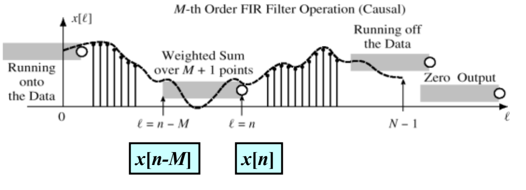
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- FILTER **ORDER** is M
- FILTER **LENGTH** is $L = M+1$
  - NUMBER of FILTER COEFFS is L

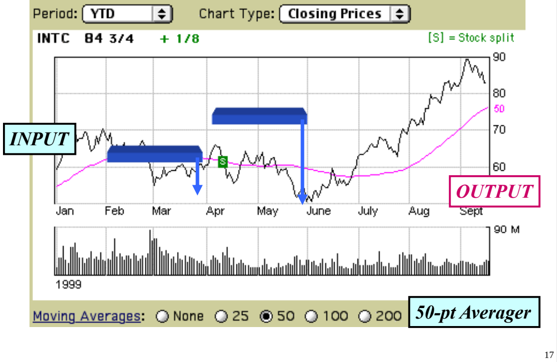
GENERAL FIR FILTER

- SLIDE a WINDOW across $x[n]$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



FILTERED STOCK SIGNAL

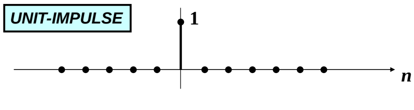


SPECIAL INPUT SIGNALS

- $x[n]$ = SINUSOID — FREQUENCY RESPONSE (LATER)
- $x[n]$ has only one NON-ZERO VALUE

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

UNIT-IMPULSE



UNIT IMPULSE SIGNAL $\delta[n]$

n	...	-2	-1	0	1	2	3	4	5	6	...
$\delta[n]$	0	0	0	1	0	0	0	0	0	0	0
$\delta[n-3]$	0	0	0	0	0	0	1	0	0	0	0

δ[n] is NON-ZERO When its argument is equal to ZERO

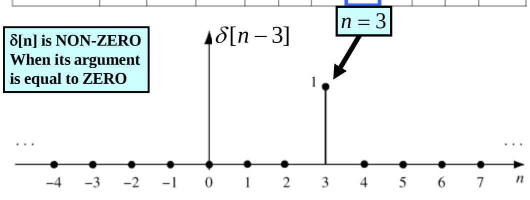
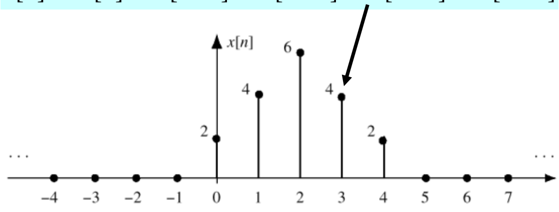


Figure 5.7 Shifted impulse sequence, $\delta[n-3]$.

MATH FORMULA for $x[n]$

- Use SHIFTED IMPULSES to write $x[n]$

$$x[n] = 2\delta[n] + 4\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$



SUM of SHIFTED IMPULSES

n	...	-2	-1	0	1	2	3	4	5	6	...
$2\delta[n]$	0	0	0	2	0	0	0	0	0	0	0
$4\delta[n-1]$	0	0	0	0	4	0	0	0	0	0	0
$6\delta[n-2]$	0	0	0	0	0	6	0	0	0	0	0
$4\delta[n-3]$	0	0	0	0	0	0	4	0	0	0	0
$2\delta[n-4]$	0	0	0	0	0	0	0	2	0	0	0
$x[n]$	0	0	0	2	4	6	4	2	0	0	0

$$x[n] = \sum_k x[k]\delta[n-k]$$

This formula ALWAYS works

$$= \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots \qquad (5.3.6)$$

4-pt AVERAGER

- CAUSAL SYSTEM: USE PAST VALUES

$$y[n] = \tfrac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$$

- INPUT = UNIT IMPULSE SIGNAL = $\boldsymbol{\delta[n]}$

$$x[n] = \delta[n]$$
$$y[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{4}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3]$$

- OUTPUT is called "IMPULSE RESPONSE"

$$h[n] = \{\ldots, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, \ldots\}$$

4-pt Avg Impulse Response

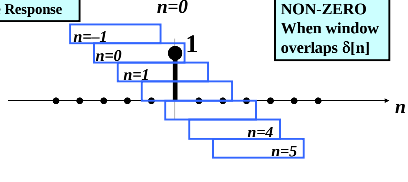
$$y[n] = \tfrac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$$

$\delta[n]$ "READS OUT" the FILTER COEFFICIENTS

$$h[n] = \{\ldots, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, \ldots\}$$

"h" in h[n] denotes Impulse Response

NON-ZERO When window overlaps δ[n]



FIR IMPULSE RESPONSE

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

n	$n<0$	0	1	2	3	...	M	$M+1$	$n>M+1$
$x[n]=\delta[n]$	0	1	0	0	0	0	0	0	0
$y[n]=h[n]$	0	b_0	b_1	b_2	b_3	...	b_M	0	0

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

FILTERING EXAMPLE

- 7-point AVERAGER $y_7[n] = \sum_{k=0}^{6} \left(\frac{1}{7}\right) x[n-k]$
  - Removes cosine
    - By making its amplitude (A) smaller
- 3-point AVERAGER $y_3[n] = \sum_{k=0}^{2} \left(\frac{1}{3}\right) x[n-k]$
  - Changes A slightly

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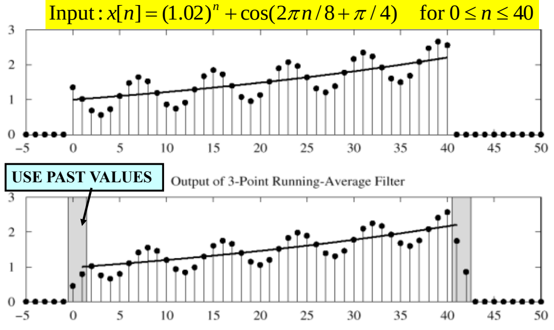
3-pt AVG EXAMPLE

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4) \quad \text{for } 0 \le n \le 40$



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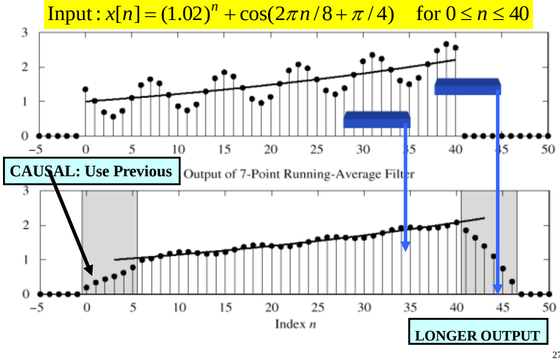
7-pt FIR EXAMPLE (AVG)

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4) \quad \text{for } 0 \le n \le 40$



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Linearity & Time-Invariance, Convolution

- IMPULSE RESPONSE, $h[n]$
  - FIR case: same as $\{b_k\}$
- CONVOLUTION
  - GENERAL: $y[n] = h[n] * x[n]$
  - GENERAL CLASS of SYSTEMS
  - LINEAR and TIME-INVARIANT
- ALL **LTI** systems have $h[n]$ & use convolution

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DIGITAL FILTERING

- CONCENTRATE on the FILTER (DSP)
- DISCRETE-TIME SIGNALS
  - FUNCTIONS of n, the "time index"
  - INPUT $x[n]$
  - OUTPUT $y[n]$

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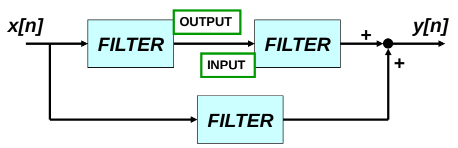
BUILDING BLOCKS



- BUILD UP COMPLICATED FILTERS
  - FROM SIMPLE MODULES
  - Ex: FILTER MODULE MIGHT BE 3-pt FIR

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GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]$$

  - For example, $b_k=\{3,-1,2,1\}$

$$y[n]=\sum_{k=0}^{3} b_k x[n-k]$$
$$=3x[n]-x[n-1]+2x[n-2]+x[n-3]$$

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MATLAB for FIR FILTER

- `yy = conv(bb,xx)`
  - VECTOR **bb** contains Filter Coefficients
    - https://www.mathworks.com/help/matlab/ref/conv.html
- FILTER COEFFICIENTS $\{b_k\}$

$$\boxed{y[n]=\sum_{k=0}^{M} b_k x[n-k]}$$

conv2() *for images*

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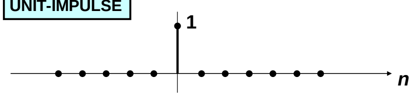
SPECIAL INPUT SIGNALS

- $x[n]$ = SINUSOID **FREQUENCY RESPONSE**
- $x[n]$ has only one NON-ZERO VALUE

$$\delta[n]=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$$

UNIT-IMPULSE



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FIR IMPULSE RESPONSE

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

n	$n<0$	0	1	2	3	$\dots$	M	$M+1$	$n>M+1$
$x[n]=\delta[n]$	0	1	0	0	0	0	0	0	0
$y[n]=h[n]$	0	b_0	b_1	b_2	b_3	$\dots$	b_M	0	0

$$\boxed{h[n]=\sum_{k=0}^{M} b_k \delta[n-k]}$$

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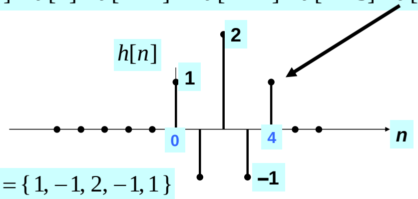
MATH FORMULA for *h*[*n*]

- Use SHIFTED IMPULSES to write $h[n]$

$$h[n]=\delta[n]-\delta[n-1]+2\delta[n-2]-\delta[n-3]+\delta[n-4]$$



$b_k=\{1,-1,2,-1,1\}$

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LTI: Convolution Sum

- **Output = Convolution of $x[n]$ & $h[n]$**
  - NOTATION: $y[n]=h[n]*x[n]$
  - Here is the FIR case:

FINITE LIMITS

$$y[n]=\sum_{k=0}^{M} h[k]x[n-k]$$

Same as b_k **FINITE LIMITS**

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CONVOLUTION Example

$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$
$x[n] = u[n]$

n	−1	0	1	2	3	4	5	6	7
$x[n]$	0	1	1	1	1	1	1	1	...
$h[n]$	0	1	−1	2	−1	1	0	0	0
$h[0]x[n]$	0	1	1	1	1	1	1	1	1
$h[1]x[n-1]$	0	0	−1	−1	−1	−1	−1	−1	−1
$h[2]x[n-2]$	0	0	0	2	2	2	2	2	2
$h[3]x[n-3]$	0	0	0	0	−1	−1	−1	−1	−1
$h[4]x[n-4]$	0	0	0	0	0	1	1	1	1
$y[n]$	0	1	0	2	1	2	2	2	...

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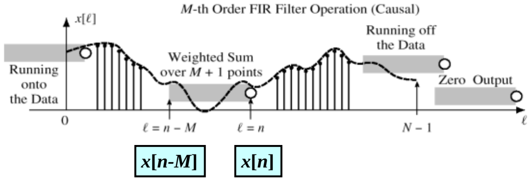
GENERAL FIR FILTER

- SLIDE a Length-L WINDOW over x[n]

$$\boxed{y[n] = \sum_{k=0}^{M} b_k x[n-k]}$$

M-th Order FIR Filter Operation (Causal)



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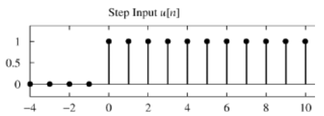
POP QUIZ

- FIR Filter is "FIRST DIFFERENCE"
  - **y[n] = x[n] - x[n-1]**
- INPUT is "UNIT STEP"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- Find $\boldsymbol{y[n]}$ $\quad y[n] = u[n] - u[n-1] = \delta[n]$

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HARDWARE STRUCTURES

x[n] → **FILTER** → y[n] $\qquad y[n] = \sum_{k=0}^{M} b_k x[n-k]$

- INTERNAL STRUCTURE of "FILTER"
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE "HOOK" THEM TOGETHER?
- SIGNAL FLOW GRAPH NOTATION

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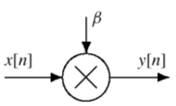
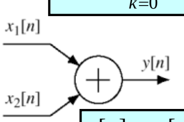
HARDWARE ATOMS

- Add, Multiply & Store

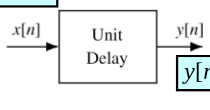
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



$y[n] = \beta x[n]$



$y[n] = x_1[n] + x_2[n]$



$y[n] = x[n-1]$

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FIR STRUCTURE

- Direct Form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

SIGNAL FLOW GRAPH

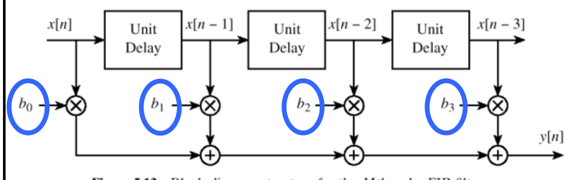

Figure 5.13 Block-diagram structure for the *M*th order FIR filter.

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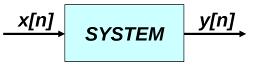
SYSTEM PROPERTIES



- MATHEMATICAL DESCRIPTION
- TIME-INVARIANCE
- LINEARITY
- CAUSALITY
  - "No output prior to input"

TIME-INVARIANCE

- IDEA:
  - "Time-Shifting the input will cause the same time-shift in the output"
- EQUIVALENTLY,
  - We can prove that
    - The time origin (n=0) is picked arbitrary

TESTING Time-Invariance

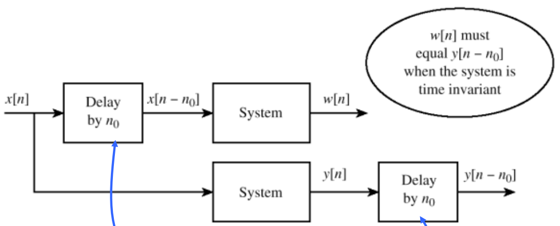


Figure 5.16 Testing time-invariance property by checking the interchange of operations.

LINEAR SYSTEM

- LINEARITY = Two Properties
- SCALING
  - "Doubling $x[n]$ will double $y[n]$"
- SUPERPOSITION:
  - "Adding two inputs gives an output that is the sum of the individual outputs"

TESTING LINEARITY

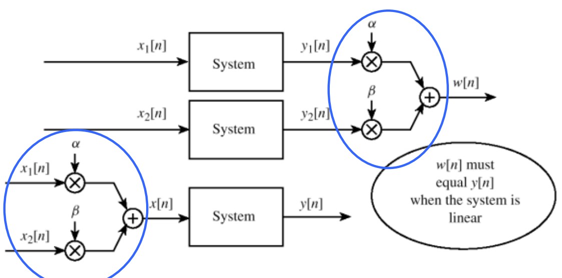


Figure 5.17 Testing linearity by checking the interchange of operations.

LTI SYSTEMS

- LTI: Linear & Time-Invariant
- COMPLETELY CHARACTERIZED by:
  - IMPULSE RESPONSE $h[n]$
  - CONVOLUTION: $y[n] = x[n]*h[n]$
    - The "rule" defining the system can ALWAYS be re-written as convolution
- FIR Example: $h[n]$ is same as b_k

POP QUIZ

- FIR Filter is "FIRST DIFFERENCE"
  - $y[n] = x[n] - x[n-1]$
- Write output as a convolution
  - Need impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

  - Then, another way to compute the output:

$$y[n] = (\delta[n] - \delta[n-1]) * x[n]$$

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CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$

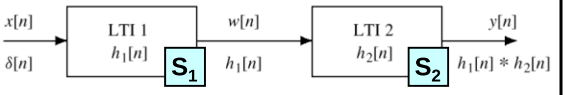

Figure 5.19 A Cascade of Two LTI Systems.

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CASCADE EQUIVALENT

- Find "overall" $h[n]$ for a cascade ?

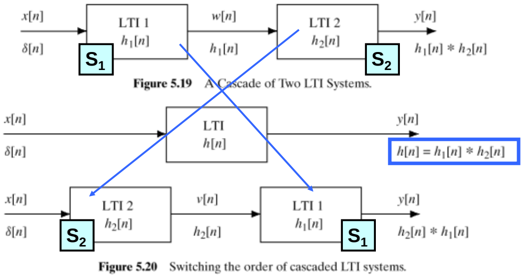

Figure 5.20 Switching the order of cascaded LTI systems.

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DOMAINS: Time & Frequency

- **Time-Domain: "n" = time**
  - $x[n]$ discrete-time signal
  - $x(t)$ continuous-time signal
- **Frequency Domain (sum of sinusoids)**
  - Spectrum vs. f (Hz)
    - ANALOG vs. DIGITAL
  - Spectrum vs. omega-hat
- Move back and forth QUICKLY

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FREQUENCY RESPONSE

- INPUT: x[n] = SINUSOID
- OUTPUT: y[n] will also be a SINUSOID
  - Different Amplitude and Phase
  - **SAME** Frequency
- AMPLITUDE & PHASE CHANGE
  - Called the **FREQUENCY RESPONSE**

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COMPLEX EXPONENTIAL

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

x[n] is the input signal—a complex exponential

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR DIFFERENCE EQUATION

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COMPLEX EXP OUTPUT

- Use the FIR "Difference Equation"

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]=\sum_{k=0}^{M} b_k A e^{j\varphi} e^{j\hat{\omega}(n-k)}$$

$$=\left(\sum_{k=0}^{M} b_k e^{j\hat{\omega}(-k)}\right) A e^{j\varphi} e^{j\hat{\omega}n}$$

$$= H(\hat{\omega}) A e^{j\varphi} e^{j\hat{\omega}n}$$

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FREQUENCY RESPONSE

- At each frequency, we can DEFINE

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

FREQUENCY RESPONSE

- Complex-valued formula
  - Has MAGNITUDE vs. frequency
  - And PHASE vs. frequency
- Notation: $H(e^{j\hat{\omega}})$ in place of $H(\hat{\omega})$

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EXAMPLE 6.1

$\{b_k\}=\{1,\ 2,\ 1\}$

$$H(e^{j\hat{\omega}})=1+2e^{-j\hat{\omega}}+e^{-j2\hat{\omega}}$$
$$=e^{-j\hat{\omega}}(e^{j\hat{\omega}}+2+e^{-j\hat{\omega}})$$
$$=e^{-j\hat{\omega}}(2+2\cos\hat{\omega})$$

EXPLOIT SYMMETRY

Since $(2+2\cos\hat{\omega})\geq 0$

Magnitude is $|H(e^{j\hat{\omega}})|=(2+2\cos\hat{\omega})$

and Phase is $\angle H(e^{j\hat{\omega}})=-\hat{\omega}$

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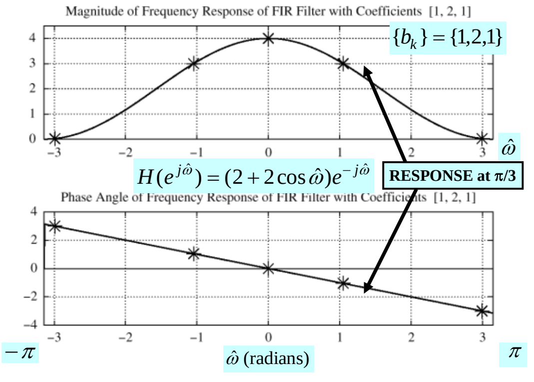
PLOT of FREQ RESPONSE



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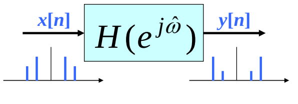
EXAMPLE 6.2

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known and $x[n]=2e^{j\pi/4}e^{j(\pi/3)n}$



$$H(e^{j\hat{\omega}})=(2+2\cos\hat{\omega})e^{-j\hat{\omega}}$$

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EXAMPLE 6.2 (answer)

Find $y[n]$ when $x[n]=2e^{j\pi/4}e^{j(\pi/3)n}$

One Step - evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega}=\pi/3$

$$H(e^{j\hat{\omega}})=(2+2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$H(e^{j\hat{\omega}})=3e^{-j\pi/3} \quad @\,\hat{\omega}=\pi/3$$

$$y[n]=\left(3e^{-j\pi/3}\right)\times 2e^{j\pi/4}e^{j(\pi/3)n}=6e^{-j\pi/12}e^{j(\pi/3)n}$$

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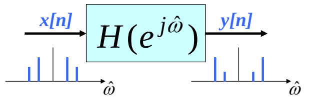
EXAMPLE: COSINE INPUT

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known and $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$



$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

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EX: COSINE INPUT

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$2\cos(\tfrac{\pi}{3}n + \tfrac{\pi}{4}) = e^{j(\pi n/3 + \pi/4)} + e^{-j(\pi n/3 + \pi/4)}$$
$$\Rightarrow x[n] = x_1[n] + x_2[n]$$

Use Linearity

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3 + \pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3 + \pi/4)}$$
$$\Rightarrow y[n] = y_1[n] + y_2[n]$$

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EX: COSINE INPUT (ans-2)

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3 + \pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3 + \pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3 + \pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3 + \pi/4)}$$

$$y[n] = 3e^{j(\pi n/3 - \pi/12)} + 3e^{-j(\pi n/3 - \pi/12)}$$
$$\Rightarrow y[n] = 6\cos(\tfrac{\pi}{3}n - \tfrac{\pi}{12})$$

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MATLAB: FREQUENCY RESPONSE

- `HH = freqz(bb,1,ww)`
  - VECTOR **bb** contains Filter Coefficients
    - https://www.mathworks.com/help/signal/ref/freqz.html
- FILTER COEFFICIENTS $\{b_k\}$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}$$

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Time & Frequency Relation

- Get Frequency Response from $h[n]$
  - Here is the FIR case:

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}}$$

IMPULSE RESPONSE

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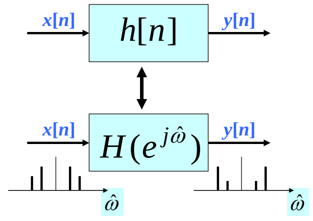
BLOCK DIAGRAMS

- Equivalent Representations



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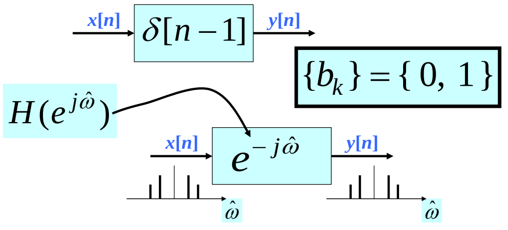
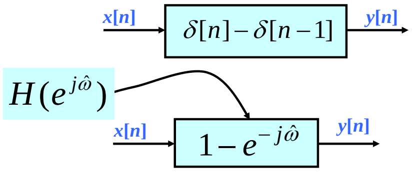
UNIT-DELAY SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-1]$



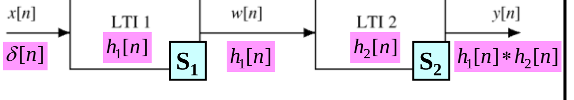
FIRST DIFFERENCE SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for the Difference Equation: $y[n] = x[n] - x[n-1]$



CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$
  - WHAT is the overall FREQUENCY RESPONSE ?



CASCADE EQUIVALENT

- MULTIPLY the Frequency Responses