BLM2041 Signals and Systems

Week 5

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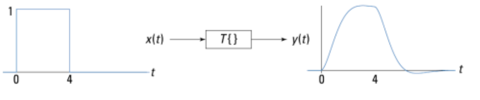
Responses to arbitrary signals

- Although we have focused on responses to simple signals (δ[n],δ(t)) we are generally interested in responses to more complicated signals.
- How do we compute responses to a more complicated input signals?



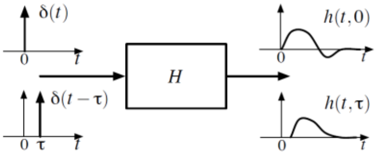
Block diagram depicting a general input/output relationship.

Impulse Response

The *impulse response* of a linear system $h_\tau(t)$ is the output of the system at time t to an impulse at time τ. This can be written as

$$h_\tau = H(\delta_\tau)$$

Care is required in interpreting this expression!



Note: Be aware of potential confusion here:

When you write

$$h_\tau(t) = H(\delta_\tau(t))$$

the variable t serves different roles on each side of the equation.

- t on the left is a specific value for time, the time at which the output is being sampled.
- t on the right is varying over all real numbers, it is not the same t as on the left.
- The output at time specific time t on the left in general depends on the input at all times t on the right (the entire input waveform).

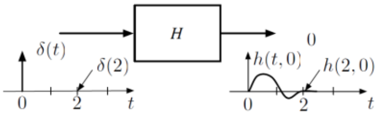
- Assume the input impulse is at $\tau = 0$,

$$h = h_0 = H(\delta_0).$$

We want to know the impulse response at time $t = 2$. It doesn't make any sense to set $t = 2$, and write

$$h(2) = H(\delta(2)) \qquad \Leftarrow \text{No!}$$

First, $\delta(2)$ is something like zero, so $H(0)$ would be zero. Second, the value of $h(2)$ depends on the entire input waveform, not just the value at $t = 2$.

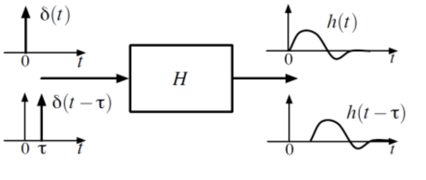


Time-invariance

If H is time invariant, delaying the input and output both by a time τ should produce the same response

$$h_\tau(t) = h(t - \tau).$$

In this case, we don't need to worry about h_τ because it is just h shifted in time.



Linearity and Extended Linearity

Linearity: A system S is linear if it satisfies both

- *Homogeneity:* If $y = Sx$, and a is a constant then
$$ay = S(ax).$$
- *Superposition:* If $y_1 = Sx_1$ and $y_2 = Sx_2$, then
$$y_1 + y_2 = S(x_1 + x_2).$$

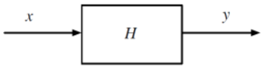
Combined Homogeneity and Superposition:
If $y_1 = Sx_1$ and $y_2 = Sx_2$, and a and b are constants,
$$ay_1 + by_2 = S(ax_1 + bx_2)$$

Extended Linearity

- *Summation:* If $y_n = S(x_n)$ for all n, an integer from $(-\infty < n < \infty)$, and a_n are constants
$$\sum_n a_n y_n = S\left(\sum_n a_n x_n\right)$$
Summation and the system operator commute, and can be interchanged.
- *Integration (Simple Example) :* If $y = S(x)$,
$$\int_{-\infty}^{\infty} a(\tau)y(t-\tau)\, d\tau = S\left(\int_{-\infty}^{\infty} a(\tau)x(t-\tau)d\tau\right)$$
Integration and the system operator commute, and can be interchanged.

Output of an LTI System

We would like to determine an expression for the output $y(t)$ of an linear time invariant system, given an input $x(t)$



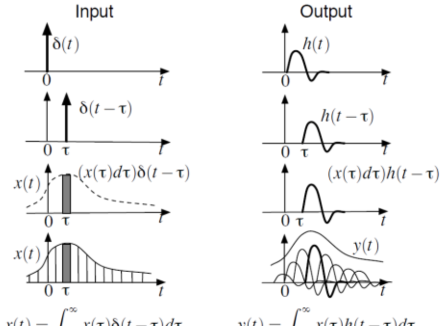
We can write a signal $x(t)$ as a sample of itself
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)\, d\tau$$
This means that $x(t)$ can be written as a weighted integral of δ functions.

Applying the system H to the input $x(t)$,
$$\begin{aligned} y(t) &= H(x(t)) \\ &= H\left(\int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)d\tau\right) \end{aligned}$$
If the system obeys extended linearity we can interchange the order of the system operator and the integration
$$y(t) = \int_{-\infty}^{\infty} x(\tau)H(\delta_\tau(t))\, d\tau.$$
The impulse response is
$$h_\tau(t) = H(\delta_\tau(t)).$$

Substituting for the impulse response gives
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau.$$
This is a *superposition integral.* The values of $x(\tau)h(t,\tau)d\tau$ are superimposed (added up) for each input time τ.

If H is time invariant, this written more simply as
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau.$$
This is in the form of a *convolution integral*, which will be the subject of the next class.

Graphically, this can be represented as:



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

$$\delta(t) \rightarrow \boxed{\text{system}} \rightarrow h(t)$$

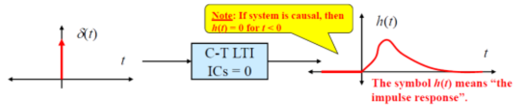
$$\delta(t-\tau) \rightarrow \boxed{\text{system}} \rightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \rightarrow \boxed{\text{system}} \rightarrow x(\tau)h(t-\tau)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \boxed{\text{system}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Recall: Impulse Response

Earlier we introduced the concept of impulse response…
…what comes out of a system when the input is an impulse (delta function)



Noting that the LT of $\delta(t) = 1$ and using the properties of the transfer function and the Z transform we said that

$h(t) = \mathscr{L}^{-1}\{H(s)\mathscr{L}\{\delta(t)\}\}$ $\boxed{h(t) = \mathscr{L}^{-1}\{H(s)\}}$ $\boxed{h(t) = \mathscr{F}^{-1}\{H(\omega)\}}$

So…once we have either $H(s)$ or $H(\omega)$ we can get the impulse response $h(t)$

Convolution Property and System Output

Let $x(t)$ be a signal with CTFT $X(\omega)$ and LT of $X(s)$ $\boxed{x(t) \leftrightarrow X(\omega)}$ $\boxed{x(t) \leftrightarrow X(s)}$

Consider a system w/ freq resp $H(\omega)$ & trans func $H(s)$ $\boxed{h(t) \leftrightarrow H(\omega)}$ $\boxed{h(t) \leftrightarrow H(s)}$

We've spent much time using these tools to analyze system outputs this way:

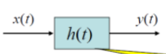
$$Y(\omega) = H(\omega)X(\omega) \quad \leftrightarrow \quad y(t) = \mathscr{F}^{-1}\{H(\omega)X(\omega)\}$$

$$Y(s) = H(s)X(s) \quad \leftrightarrow \quad y[n] = \mathscr{L}^{-1}\{H(s)X(s)\}$$

The convolution property of the CTFT and LT gives an alternate way to find $y(t)$:

$$\boxed{\mathscr{F}^{-1}\{X(\omega)H(\omega)\} = x(t) * h(t)} \qquad \boxed{x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

$$\boxed{\mathscr{L}^{-1}\{X(s)H(s)\} = x(t) * h(t)}$$



$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

"Convolving" input $x(t)$ with the impulse response $h(t)$ gives the output $y(t)$!

LTI System with impulse response $h(t)$

Convolution for Causal System & with Causal Input

An arbitrary LTI system's output can be found using the general convolution form:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau} \qquad \textbf{General LTI System}$$

If the system is causal then $h(t) = 0$ for $t < 0$…. Thus $h(t-\tau) = 0$ for $t > \tau$… so:

$$\boxed{y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau} \qquad \textbf{Causal LTI System}$$

If the input is causal then $x(t) = 0$ for $t < 0$…. so:

$$\boxed{y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau} \qquad \textbf{Causal Input \& General LTI System}$$

If the system & signal are both causal then

$$\boxed{y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau} \qquad \textbf{Causal Input \& Causal LTI System}$$

Convolution Properties

1. **Commutativity** $\boxed{x(t) * h(t) = h(t) * x(t)}$

2. **Associativity** $\boxed{[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]}$

Associativity together with commutativity says we **can interchange the order of two cascaded systems**.

3. **Distributivity** $\boxed{x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)}$

4. **Derivative Property:**

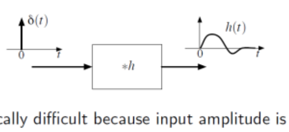
$$\frac{d}{dt}[x(t) * v(t)] = \dot{x}(t) * v(t) \quad \text{derivative}$$
$$= x(t) * \dot{v}(t)$$

5. **Integration Property** Let $y(t) = x(t) * h(t)$, then

$$\int_{-\infty}^{t} y(\lambda)d\lambda = \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right] * h(t) = x(t) * \left[\int_{-\infty}^{t} h(\lambda)d\lambda\right]$$

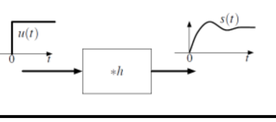
Example: Measuring the impulse response of an LTI system.

We would like to measure the impulse response of an LTI system, described by the impulse response $h(t)$

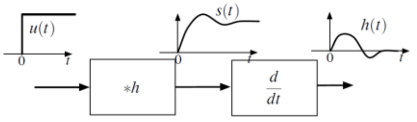


This can be practically difficult because input amplitude is often limited. A very short pulse then has very little energy.

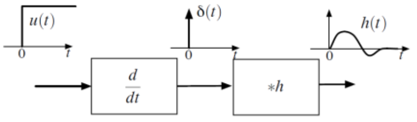
A common alternative is to measure the *step response* $s(t)$, the response to a unit step input $u(t)$



The impulse response is determined by differentiating the step response,



To show this, commute the convolution system and the differentiator to produce a system with the same overall impulse response



Steps for Graphical Convolution $x(t)*h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

1. **Re-Write the signals as functions of τ**: $x(\tau)$ and $h(\tau)$

2. **Flip** just one of the signals around $t = 0$ to get either $x(-\tau)$ or $h(-\tau)$
   a. It is usually best to flip the signal with shorter duration
   b. For notational purposes here: we'll flip $h(\tau)$ to get $h(-\tau)$

3. **Find Edges** of the flipped signal
   a. Find the left-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{L,0}$**
   b. Find the right-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{R,0}$**

4. **Shift** $h(-\tau)$ by an arbitrary value of t to get $h(t - \tau)$ and **get its edges**
   a. Find the left-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{L,t}$**
      - **Important:** It will *always* be… $\tau_{L,t} = t + \tau_{L,0}$

   b. Find the right-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{R,t}$**
      - **Important:** It will *always* be… $\tau_{R,t} = t + \tau_{R,0}$

   Note: If the signal you flipped is NOT finite duration, one or both of $\tau_{L,t}$ and $\tau_{R,t}$ will be infinite ($\tau_{L,t} = -\infty$ and/or $\tau_{R,t} = \infty$)

Steps Continued

5. **Find Regions of τ-Overlap**
   a. What you are trying to do here is find intervals of t over which the product $x(\tau)\, h(t - \tau)$ has a single mathematical form in terms of τ
   b. In each region find: Interval of t that makes the identified overlap happen
   c. Working examples is the best way to learn how this is done

Tips: Regions should be contiguous with no gaps!!!
Don't worry about $<$ vs. $\leq$ etc.

6. For Each Region: **Form the Product $x(\tau)\, h(t - \tau)$ and Integrate**
   a. Form product $x(\tau)\, h(t - \tau)$
   b. Find the Limits of Integration by finding the interval of τ over which the product is nonzero
      i. Found by seeing where the edges of $x(\tau)$ and $h(t - \tau)$ lie
      ii. Recall that the edges of $h(t - \tau)$ are $\tau_{L,t}$ and $\tau_{R,t}$, which often depend on the value of t
         - So… the limits of integration may depend on t
   c. Integrate the product $x(\tau)\, h(t - \tau)$ over the limits found in 6b
      i. The result is generally a function of t, but is only valid for the interval of t found for the current region
      ii. Think of the result as a "time-section" of the output $y(t)$

Steps Continued

7. **"Assemble" the output** from the output time-sections for all the regions
   a. Note: you do NOT add the sections together
   b. You define the output "piecewise"
   c. Finally, if possible, look for a way to write the output in a simpler form

Example: Graphically Convolve Two Signals

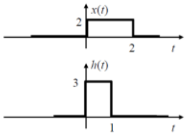
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

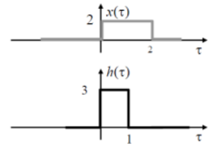
By "Properties of Convolution"… these two forms are Equal

This is why we can flip either signal

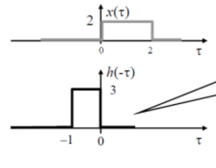
Convolve these two signals:



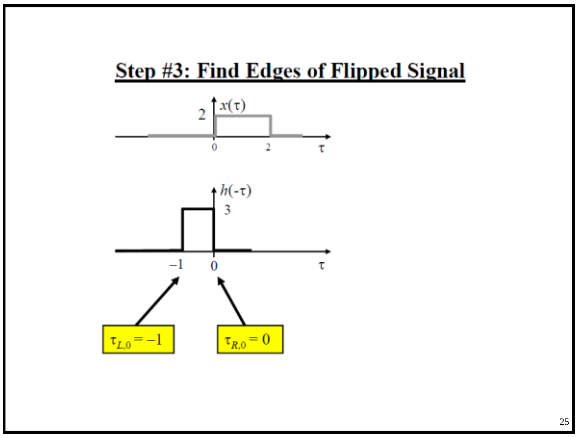
Step #1: Write as Function of τ

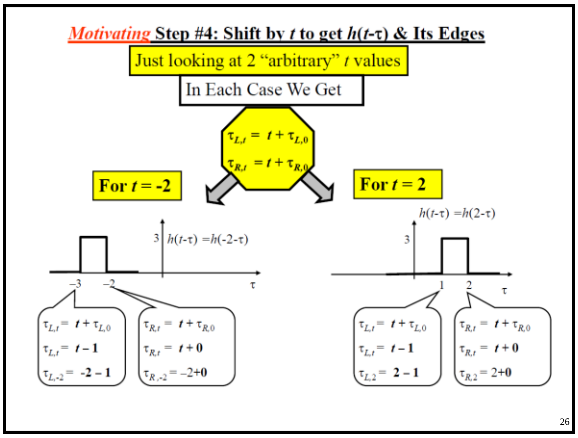


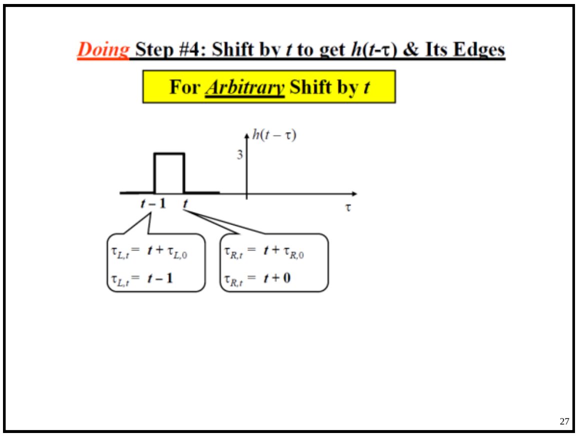
Step #2: Flip $h(\tau)$ to get $h(-\tau)$

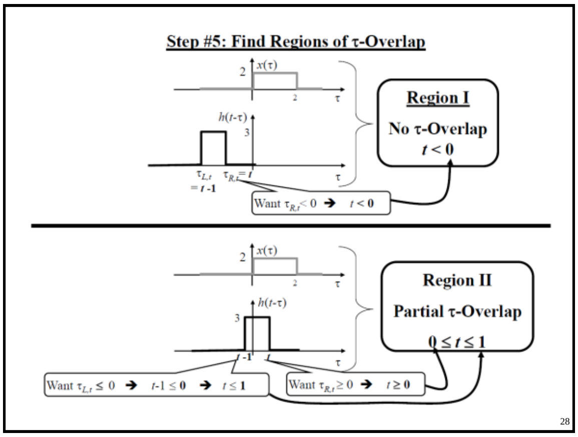


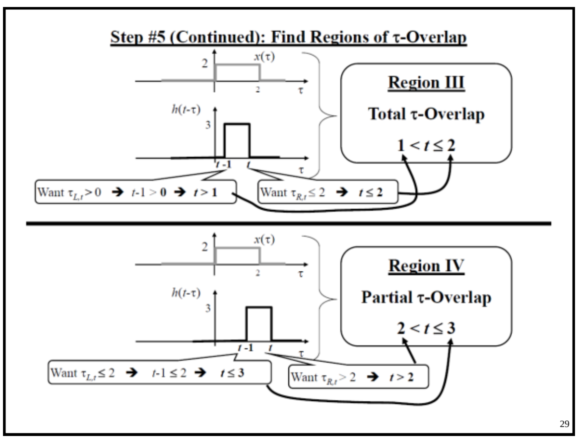
Usually Easier to Flip the Shorter Signal

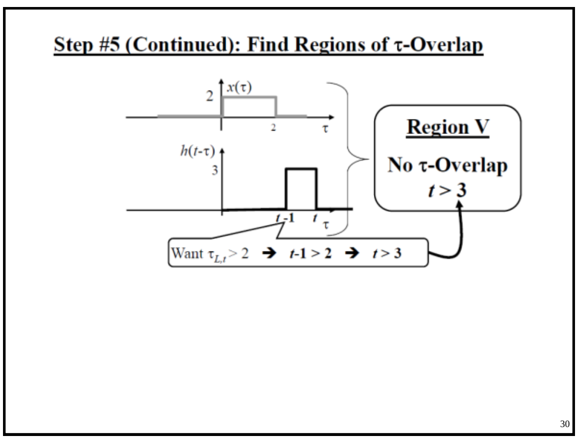

Step #3: Find Edges of Flipped Signal
x(τ)
2
0
2
τ
h(-τ)
3
–1
0
τ
τ_{L,0} = –1
τ_{R,0} = 0
25


Motivating Step #4: Shift by t to get h(t-τ) & Its Edges
Just looking at 2 "arbitrary" t values
In Each Case We Get
τ_{L,t} = t + τ_{L,0}
τ_{R,t} = t + τ_{R,0}
For t = -2
For t = 2
h(t-τ) =h(-2-τ)
h(t-τ) =h(2-τ)
3
–3
–2
1
2
τ
τ_{L,t} = t + τ_{L,0}
τ_{L,t} = t – 1
τ_{L,-2} = -2 – 1
τ_{R,t} = t + τ_{R,0}
τ_{R,t} = t + 0
τ_{R,-2} = –2+0
τ_{L,t} = t + τ_{L,0}
τ_{L,t} = t – 1
τ_{L,2} = 2 – 1
τ_{R,t} = t + τ_{R,0}
τ_{R,t} = t + 0
τ_{R,2} = 2+0
26


Doing Step #4: Shift by t to get h(t-τ) & Its Edges
For Arbitrary Shift by t
h(t – τ)
3
t – 1
t
τ
τ_{L,t} = t + τ_{L,0}
τ_{L,t} = t – 1
τ_{R,t} = t + τ_{R,0}
τ_{R,t} = t + 0
27

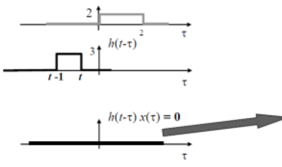

Step #5: Find Regions of τ-Overlap
x(τ)
2
2
τ
h(t-τ)
3
τ_{L,t} = t -1
τ_{R,t} = t
Want τ_{R,t} < 0 ➔ t < 0
Region I
No τ-Overlap
t < 0
x(τ)
h(t-τ)
t -1
t
Want τ_{L,t} ≤ 0 ➔ t-1 ≤ 0 ➔ t ≤ 1
Want τ_{R,t} ≥ 0 ➔ t ≥ 0
Region II
Partial τ-Overlap
0 ≤ t ≤ 1
28


Step #5 (Continued): Find Regions of τ-Overlap
x(τ)
2
2
τ
h(t-τ)
3
t -1
t
Want τ_{L,t} > 0 ➔ t-1 > 0 ➔ t > 1
Want τ_{R,t} ≤ 2 ➔ t ≤ 2
Region III
Total τ-Overlap
1 < t ≤ 2
x(τ)
h(t-τ)
t -1
t
Want τ_{L,t} ≤ 2 ➔ t-1 ≤ 2 ➔ t ≤ 3
Want τ_{R,t} > 2 ➔ t > 2
Region IV
Partial τ-Overlap
2 < t ≤ 3
29


Step #5 (Continued): Find Regions of τ-Overlap
x(τ)
2
2
τ
h(t-τ)
3
t -1
t
τ
Want τ_{L,t} > 2 ➔ t-1 > 2 ➔ t > 3
Region V
No τ-Overlap
t > 3
30

Step #6: Form Product & Integrate For Each Region

Region I: $t < 0$

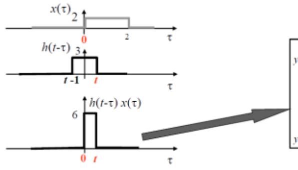


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} 0\,d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t < 0$$

With 0 integrand the limits don't matter!!!

Region II: $0 \le t \le 1$

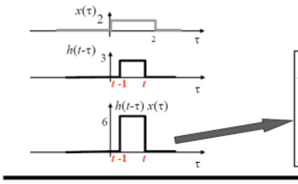


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 6\,d\tau = [6\tau]_0^t = 6t - 6\times 0 = 6t$$

$$y(t) = 6t \quad \text{for} \quad 0 \le t \le 1$$

Step #6 (Continued): Form Product & Integrate For Each Region

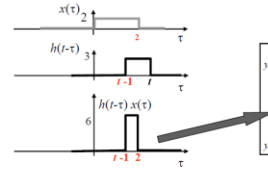
Region III: $1 < t \le 2$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} 6\,d\tau = [6\tau]_{t-1}^{t} = 6t - 6(t-1) = 6$$

$$y(t) = 6 \quad \text{for all } t \text{ such that: } 1 < t \le 2$$

Region IV: $2 < t \le 3$

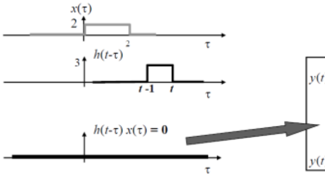


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{2} 6\,d\tau = [6\tau]_{t-1}^{2} = 6\times 2 - 6(t-1) = -6t + 18$$

$$y(t) = -6t + 18 \quad \text{for} \quad 2 < t \le 3$$

Step #6 (Continued): Form Product & Integrate For Each Region

Region V: $t > 3$

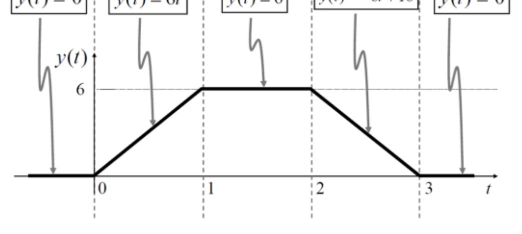


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} 0\,d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t > 3$$

Step #7: "Assemble" Output Signal

Region I	Region II	Region III	Region IV	Region V
$t < 0$	$0 \le t \le 1$	$1 < t \le 2$	$2 < t \le 3$	$t > 3$
$y(t) = 0$	$y(t) = 6t$	$y(t) = 6$	$y(t) = -6t + 18$	$y(t) = 0$



Discrete Convolution

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \rightarrow \boxed{\text{system}} \rightarrow h[n]$$

$$\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow h[n-k]$$

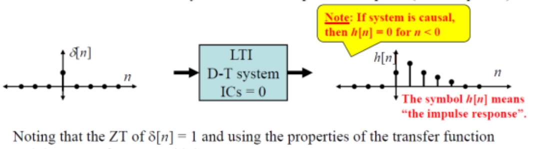
$$x[k]\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete Convolution

Recall: Impulse Response

Earlier we introduced the concept of impulse response…
…what comes out of a system when the input is an impulse (delta sequence)



Noting that the ZT of $\delta[n] = 1$ and using the properties of the transfer function and the Z transform we said that

$$h[n] = Z^{-1}\{H(z)Z\{\delta[n]\}\} \qquad \boxed{h[n] = Z^{-1}\{H(z)\}} \qquad \boxed{h[n] = IDTFT\{H(\Omega)\}}$$

So…once we have either $H(z)$ or $H(\Omega)$ we can get the impulse response $h[n]$

Convolution Property and System Output

$x[n] \leftrightarrow X(\Omega)$
$x[n] \leftrightarrow X(z)$

Let $x[n]$ be a signal with DTFT $X(\Omega)$ and ZT of $X(z)$

$h[n] \leftrightarrow H(\Omega)$
$h[n] \leftrightarrow H(z)$

Consider a system w/ freq resp $H(\Omega)$ & trans func $H(z)$

We've spent much time using these tools to analyze system outputs this way:

$$Y(\Omega) = H(\Omega)X(\Omega) \quad \leftrightarrow \quad y[n] = DTFT^{-1}\{H(\Omega)X(\Omega)\}$$

$$Y(z) = H(z)X(z) \quad \leftrightarrow \quad y[n] = Z^{-1}\{H(z)X(z)\}$$

The convolution property of the DTFT and ZT gives an alternate way to find $y[n]$:

$$DTFT^{-1}\{X(\Omega)H(\Omega)\} = x[n]*h[n]$$

$$Z^{-1}\{X(z)H(z)\} = x[n]*h[n]$$

$$x[n]*h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

"Convolving" input $x[n]$ with the impulse response $h[n]$ gives the output $y[n]$!

LTI System with impulse response $h[n]$

37

Convolution for *Causal* System & with *Causal* Input

An arbitrary LTI system's output can be found using the general convolution form:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$ **General LTI System**

If the system is causal then $h[n] = 0$ for $n < 0$… Thus $h[n-m] = 0$ for $m > n$ … so:

$$y[n] = \sum_{m=-\infty}^{n} x[m]h[n-m]$$ **Causal LTI System**

If the input is causal then $x[n] = 0$ for $n < 0$…. so:

$$y[n] = \sum_{m=0}^{\infty} x[m]h[n-m]$$ **Causal Input & General LTI System**

If the system & signal are both causal then

$$y[n] = \sum_{m=0}^{n} x[m]h[n-m]$$ **Causal Input & Causal LTI System**

38

Convolution Properties (can sometimes exploit to make things easier)

1. Commutativity

$$x[n]*h[n] = h[n]*x[n]$$

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \quad = \quad h[n] \rightarrow \boxed{x[n]} \rightarrow y[n]$

This is obvious from the frequency domain (or z domain) viewpoint:

$$x[n]*h[n] = h[n]*x[n] \Rightarrow X(\Omega)H(\Omega) = H(\Omega)X(\Omega)$$

2. Associativity

$$(x[n]*h_1[n])*h_2[n] = x[n]*(h_1[n]*h_2[n])$$

$\Rightarrow$ Can combine cascade into single equivalent system

$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow \quad = \quad x[n] \rightarrow \boxed{h_1[n]*h_2[n]} \rightarrow$

This is obvious from the frequency domain (or z domain) viewpoint:

$$[X(\Omega)H_1(\Omega)]H_2(\Omega) = X(\Omega)\underbrace{[H_1(\Omega)H_2(\Omega)]}$$

Tells us what the Freq Resp is for a cascade

39

Associativity together with commutativity says we **can interchange the order of two cascaded systems**:

$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow \quad = \quad x[n] \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_1[n]} \rightarrow$

Warning: This holds in theory but in practice there may be physical issues that prevent this!!!

3. Distributivity

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$$

$\Rightarrow$ can combine sum of two outputs into a single system (or vice versa)

$x[n] \rightarrow \boxed{h_1[n]}, \boxed{h_2[n]} \rightarrow \Sigma \rightarrow \quad = \quad x[n] \rightarrow \boxed{h_1[n]+h_2[n]} \rightarrow$

With commutativity this says we can split a complicated input into sum of simple ones… which is nothing more than "linearity"!!

40

Graphical Convolution – To Visualize & Test Real Systems

Can do convolution this way when signals are know numerically or by equation

- Convolution involves the sum of a product of two signals: $x[i]h[n-i]$
- At each output index n, the product changes

"Commutativity" says we can flip either $x[i]$ or $h[i]$ and get the same answer

Step 1: Write both as functions of i: $x[i]$ & $h[i]$

Step 2: Flip $h[i]$ to get $h[-i]$ (The book calls this "fold")

Repeat for each n

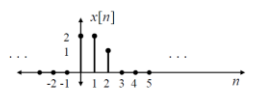
Step 3: For each output index n value of interest, shift by n to get $h[n - i]$

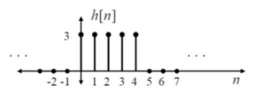
(Note: positive n gives right shift!!!!)

Step 4: Form product $x[i]h[n-i]$ and sum its elements to get the number $y[n]$

41

Example of Graphical Convolution





Find $y[n]=x[n]*h[n]$ for all integer values of n

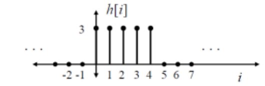
Solution

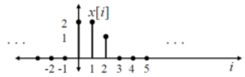
For this problem I choose to flip $x[n]$

My personal preference is to flip the shorter signal although I sometimes don't follow that "rule"… only through lots of practice can you learn how to best choose which one to flip.

42

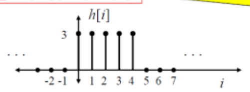
Step 1: Write both as functions of i: $x[i]$ & $h[i]$

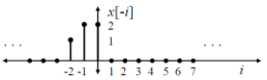




Step 2: Flip $x[i]$ to get $x[-i]$

"**Commutativity**" says we can flip either $x[i]$ or $h[i]$ and get the same answer… **Here I flipped $x[i]$**



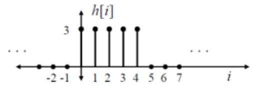


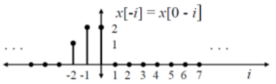
We want a solution for $n = \ldots$ -2, -1, 0, 1, 2, … so must do Steps 3&4 for all n.

But… let's first do: **Steps 3&4 for $n = 0$** and then proceed from there.

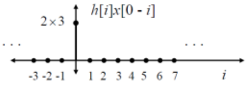
Step 3: For $n = 0$, shift by n to get $x[n - i]$

For $n = 0$ case there is no shift! $x[0 - i] = x[-i]$





Step 4: For $n = 0$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

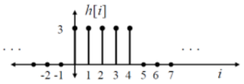


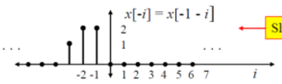
Sum over $i \Rightarrow$ $\boxed{y[0] = 6}$

Steps 3&4 for all $n < 0$

Step 3: For $n < 0$, shift by n to get $x[n - i]$

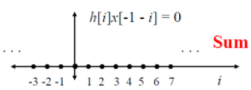
Negative n gives a left-shift





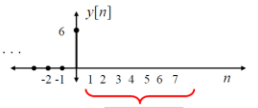
Shown here for $n = -1$

Step 4: For $n < 0$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $\boxed{y[n] = 0 \quad \forall n < 0}$

So… what we know so far is that: $\boxed{y[n] = \begin{cases} 0, & \forall n < 0 \\ 6, & n = 0 \end{cases}}$



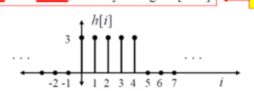
$y[n]$ = ???

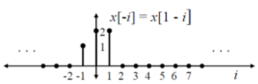
So now we have to do Steps 3&4 for $n > 0$…

Steps 3&4 for $n = 1$

Step 3: For $n = 1$, shift by n to get $x[n - i]$

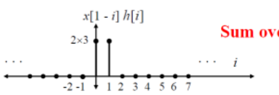
Positive n gives a Right-shift





shifted to the right by one

Step 4: For $n = 1$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

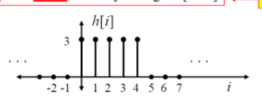


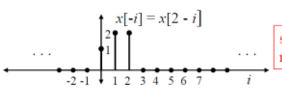
Sum over $i \Rightarrow$ $\boxed{y[1] = 6 + 6 = 12}$

Steps 3&4 for $n = 2$

Step 3: For $n = 2$, shift by n to get $x[n - i]$

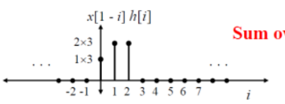
Positive n gives a Right-shift





shifted to the right by two

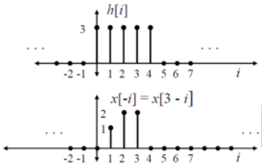
Step 4: For $n = 2$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $\boxed{y[2] = 3 + 6 + 6 = 15}$

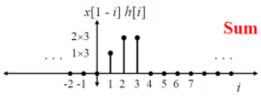
Steps 3&4 for $n = 3$

Step 3: For $n = 3$, shift by n to get $x[n - i]$ ← Positive n gives a Right-shift



shifted to the right by three

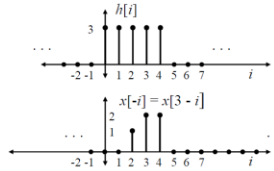
Step 4: For $n = 3$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $y[3] = 3 + 6 + 6 = 15$

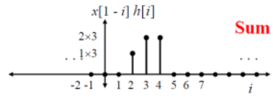
Steps 3&4 for $n = 4$

Step 3: For $n = 4$, shift by n to get $x[n - i]$ ← Positive n gives a Right-shift



shifted to the right by four

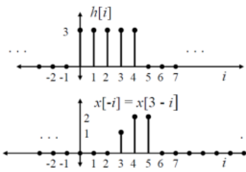
Step 4: For $n = 4$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $y[4] = 3 + 6 + 6 = 15$

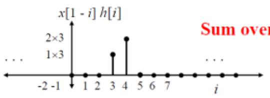
Steps 3&4 for $n = 5$

Step 3: For $n = 5$, shift by n to get $x[n - i]$ ← Positive n gives a Right-shift



shifted to the right by five

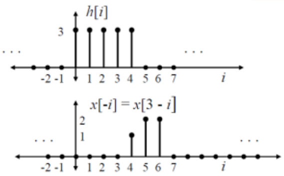
Step 4: For $n = 5$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $y[5] = 3 + 6 = 9$

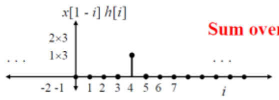
Steps 3&4 for $n = 6$

Step 3: For $n = 6$, shift by n to get $x[n - i]$ ← Positive n gives a Right-shift



shifted to the right by six

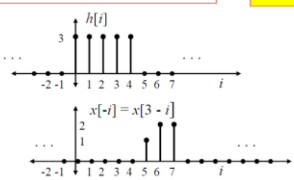
Step 4: For $n = 6$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow$ $y[6] = 3$

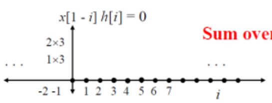
Steps 3&4 for all $n > 6$

Step 3: For $n > 6$, shift by n to get $x[n - i]$ ← Positive n gives a Right-shift



shifted to the right by seven

Step 4: For $n > 6$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

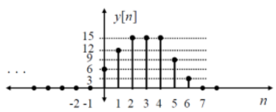


Sum over $i \Rightarrow$ $y[n] = 0 \quad \forall n > 6$

So… now we know the values of $y[n]$ for all values of n

We just need to put it all together as a function…

Here it is easiest to just plot it… you could also list it as a table.



Note that convolving these kinds of signals gives a "ramp-up" at the beginning and a "ramp-down" at the end.

Various kinds of "transients" at the beginning and end of a convolution are common.