# Q1

Plot the following cosine signal.

$$x(t) = 7\cos(0.2\pi t + 0.5\pi) \qquad -5 \le t \le 15$$

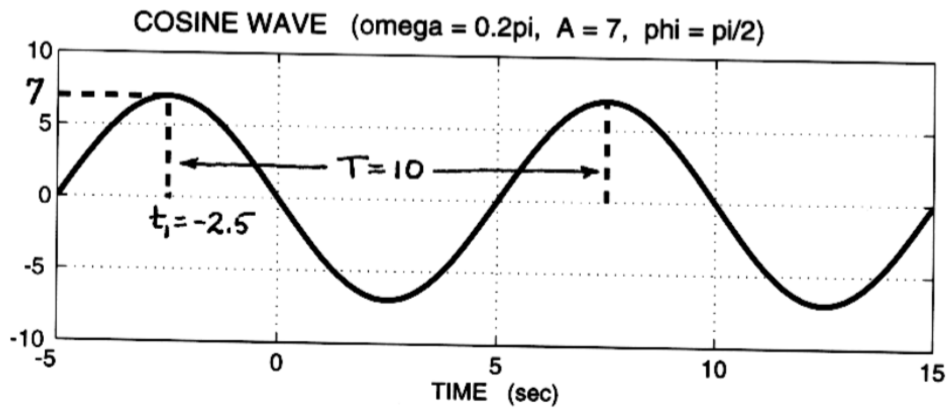
Label the axes *in detail.* In addition, determine the period of the signal.

$$x(t) = 7\cos(0.2\pi t + 0.5\pi) = 7\cos\left(2\pi\left(\tfrac{1}{10}\right)t + \tfrac{\pi}{2}\right).$$

$$\therefore\ f = \tfrac{1}{10}\,\text{Hz} \Rightarrow T = 10\,\text{sec (period)}.$$

$$\varphi = \pi/2 \Rightarrow t_1 = -\frac{\varphi}{2\pi f} = \frac{-\pi/2}{2\pi/10} = -\frac{10}{4} = -2.5\,\text{sec}$$

(LOCATION OF A POSITIVE PEAK)



# Q2

A signal $x(t)$ is defined by: $x(t) = \Re e\{(1+j)e^{j\pi t}\}$. Its shortest period ($T$) is

$$\omega = \pi \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \text{ sec}$$

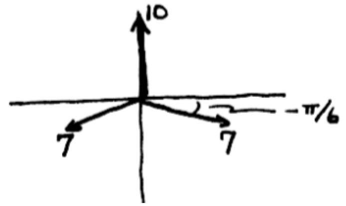
A sinusoidal signal $x(t)$ is defined by: $x(t) = \Re e\{(1+j)e^{j\pi t}\}$. When plotted versus time ($t$), its maximum value will be:

$$x(t) = \Re e\{\sqrt{2}\, e^{j\pi/4} e^{j\pi t}\}$$

$$= \underbrace{\sqrt{2}}_{A} \cos(\pi t + \pi/4)$$

Determine the amplitude ($A$) and phase ($\phi$) of the sinusoid that is the sum of the following three sinusoids: $10\cos(6t + \pi/2) + 7\cos(6t - \pi/6) + 7\cos(6t + 7\pi/6)$,

$$10e^{j\pi/2} + 7e^{-j\pi/6} + 7e^{j7\pi/6} = 10j$$




$$+7\frac{\sqrt{3}}{2} - j7/2$$

$$-7\frac{\sqrt{3}}{2} - j7/2$$

$$= j3 = 3e^{j\pi/2}$$

# Q3

Define $x(t)$ as

$$x(t) = 5\sqrt{2}\cos(20\pi t + \pi/4) + A\cos(20\pi t + \phi) \tag{1}$$

where $A$ is a *positive* number. In addition, assume that $x(t)$ has a phase of zero, so that it may be written as

$$x(t) = B\cos(20\pi t), \tag{2}$$

where $B$ is a *positive* number.

(a) What relationship must exist between $A$ and $\phi$ in order for $x(t)$ to have zero phase as indicated in Eq. 2?

(b) If $B = 10$, what are the values for $A$ and $\phi$?

# Q3 Solution

$$x(t) = 5\sqrt{2}\cos\left(20\pi t + \frac{\pi}{4}\right) + A\cos(2\pi t + \phi)$$

$$= B\cos(20\pi t)$$

$$A > 0$$
$$B > 0$$

Let $\underline{X}$ be the phasor representing $x(t)$

$$\underline{X} = 5\sqrt{2}\, e^{j\pi/4} + A e^{j\phi} = B$$

$$= 5\sqrt{2}\cos\frac{\pi}{4} + j5\sqrt{2}\sin\frac{\pi}{4} + A\cos\phi + jA\sin\phi = B$$

$$= \left(5\sqrt{2}\,\frac{\sqrt{2}}{2} + A\cos\phi\right) + j\left(5\sqrt{2}\,\frac{\sqrt{2}}{2} + A\sin\phi\right) = B$$

a) Thus: for zero phase: $\begin{cases} \text{Imaginary part zero} \\ \text{Real part positive} \end{cases}$

$$\boxed{\begin{aligned} 5 + A\sin\phi &= 0 \\ 5 + A\cos\phi &> 0 \end{aligned}}$$

b) For $B = 10$, solve: $\begin{cases} 5 + A\sin\phi = 0 \\ 5 + A\cos\phi = 10 \end{cases} \Rightarrow \begin{aligned} A\sin\phi &= -5 \\ A\cos\phi &= 5 \end{aligned}$

$$\Rightarrow \underbrace{A^2\sin^2\phi + A^2\cos^2\phi}_{=A^2} = (-5)^2 + (5)^2 = 25 + 25 = 50 \Rightarrow \boxed{A = \sqrt{50}}$$

and $\left.\begin{aligned} \cos\phi &= \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \\ \sin\phi &= -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}} \end{aligned}\right\} \rightarrow \boxed{\phi = -\pi/4}$

# Q4

The phase of a sinusoid can be related to time shift:

$$x(t) = A\cos(2\pi f_\circ t + \phi) = A\cos(2\pi f_\circ (t - t_1))$$

In the following parts, assume that the period of the sinusoidal wave is $T = 10$ sec.

(a) "When $t_1 = -2$ sec, the value of the phase is $\phi = \pi/5$."
Explain whether this is TRUE or FALSE.

(b) "When $t_1 = 5$ sec, the value of the phase is $\phi = \pi$."
Explain whether this is TRUE or FALSE.

(c) "When $t_1 = 8$ sec, the value of the phase is $\phi = 2\pi/5$."
Explain whether this is TRUE or FALSE.

# Q4 Solution

$\varphi = -2\pi f_0 t_1 \qquad T = 10\,\text{sec} \qquad f_0 = \frac{1}{T} = \frac{1}{10}\,\text{Hz}$

(a) $t_1 = -2\,\text{sec} \Rightarrow \varphi = -2\pi\left(\frac{t_1}{T}\right) = -2\pi\frac{(-2)}{10} = \frac{4\pi}{10} = \frac{2\pi}{5}$

$\boxed{\text{FALSE}}$ $\varphi \neq \pi/5$

(b) $t_1 = 5\,\text{sec} \Rightarrow \varphi = -2\pi\left(\frac{5}{10}\right) = -\pi$

But $2\pi$ radians can be added to the phase without changing the result.

$A\cos(2\pi f_0 t - \pi) = A\cos(2\pi f_0 t + \pi)$

so $\varphi = \pi$ ALSO $\Rightarrow$ $\boxed{\text{TRUE}}$

(c) $t_1 = 8\,\text{sec} \Rightarrow \varphi = -2\pi\left(\frac{8}{10}\right) = -\frac{16\pi}{10} = -\frac{8\pi}{5}$

Again, adding $2\pi$ is OK.

$A\cos\left(2\pi f_0 t - \frac{8\pi}{5}\right) = A\cos\left(2\pi f_0 t + \frac{2\pi}{5}\right)$

so, $\varphi = 2\pi/5$
$\varphi = -8\pi/5$ $\therefore$ $\boxed{\text{TRUE}}$

BOTH ARE CORRECT BECAUSE THEY DIFFER BY $2\pi$!

# Q5

Simplify the following and give the answer as a single sinusoid. Draw the vector diagram of the complex amplitudes (phasors) to show how you obtained the answer.

(a) $x_a(t) = \sqrt{2}\cos(2\pi t + 3\pi/4) - \cos(2\pi t + \pi/4)$

(b) $x_b(t) = \cos(11t + 17\pi) + \sqrt{3}\cos(11t + \pi/3) + \sqrt{3}\cos(11t - \pi/3)$

(c) $x_c(t) = \cos(\pi t + 3\pi/4) + \cos(\pi t + 5\pi/4) + \cos(\pi t - \pi/4) + 2\cos(\pi t + \pi/4)$

NOTE 1: $A\cos(\omega t+\theta) = \mathrm{Re}\{Ae^{j(\omega t+\theta)}\}$

$$= \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

EITHER WORKS

NOTE 2: IF WE ADD COS TERMS AND ALL ARE AT THE SAME FREQUENCY, THEN WE ONLY NEED TO ADD THEIR COMPLEX MAGNITUDES.

# Q5 Solution

a)

$$x_a(t) = \sqrt{2}\cos\left(2\pi t + \tfrac{3}{4}\pi\right) - \cos\left(2\pi t + \tfrac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} e^{j(2\pi t + \frac{3}{4}\pi)} + \frac{\sqrt{2}}{2} e^{-j(2\pi t + \frac{3}{4}\pi)} - e^{j(2\pi t + \frac{\pi}{4})} - e^{-j(2\pi t + \frac{\pi}{4})}$$

$$= \frac{\sqrt{2}}{2} e^{j\frac{3}{4}\pi} e^{j2\pi t} + \frac{\sqrt{2}}{2} e^{-j\frac{3}{4}\pi} e^{-j2\pi t} - e^{j\frac{\pi}{4}} e^{j2\pi t} - e^{-j\frac{\pi}{4}} e^{-j2\pi t}$$

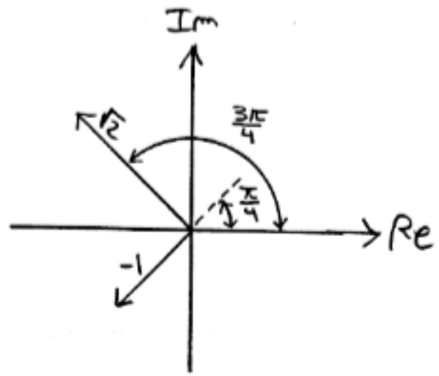
$$= \underbrace{\left[\frac{\sqrt{2}}{2} e^{j\frac{3}{4}\pi} - e^{j\frac{\pi}{4}}\right]} e^{j2\pi t} + \underbrace{\left[\frac{\sqrt{2}}{2} e^{-j\frac{3}{4}\pi} - e^{-j\frac{\pi}{4}}\right]} e^{-j2\pi t}$$

CONVERT TO RECTANGULAR AND ADD

$$x_a(t) = 0.866\, e^{j2.972} e^{j2\pi t} + 0.866\, e^{-j2.972} e^{-j2\pi t}$$

$$= 1.732\cos(2\pi t + 2.972)$$

radians



# Q5 Solution

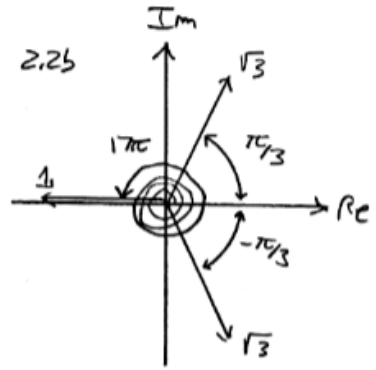
b) $x_b(t) = \cos(11t + 17\pi) + \sqrt{3}\cos(11t + \pi/3) + \sqrt{3}\cos(11t - \pi/3)$

$= Re\left[\left(e^{j17\pi} + \sqrt{3}e^{j\frac{\pi}{3}} + \sqrt{3}e^{-j\frac{\pi}{3}}\right)\left(e^{j11t}\right)\right]$

$= Re\left[0.732\, e^{j0}\, e^{j11t}\right]$

$= 0.732\cos(11t)$

2.2b



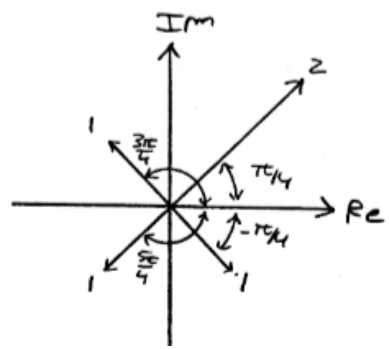

c) $x_c(t) = \cos(\pi t + 3\pi/4) + \cos(\pi t + 5\pi/4) + \cos(\pi t - \pi/4) + 2\cos(\pi t + \pi/4)$

$= Re\left[e^{j(\pi t + \frac{3}{4}\pi)} + e^{j(\pi t + \frac{5\pi}{4})} + e^{j(\pi t - \frac{\pi}{4})} + 2e^{j(\pi t + \frac{\pi}{4})}\right]$

$= Re\left[\left(e^{j\frac{3\pi}{4}} + e^{j\frac{5\pi}{4}} + e^{-\frac{\pi}{4}} + 2e^{j\frac{\pi}{4}}\right)\left(e^{j\pi t}\right)\right]$

$= Re\left[\left(e^{j0.7854}\right)\left(e^{j\pi t}\right)\right]$

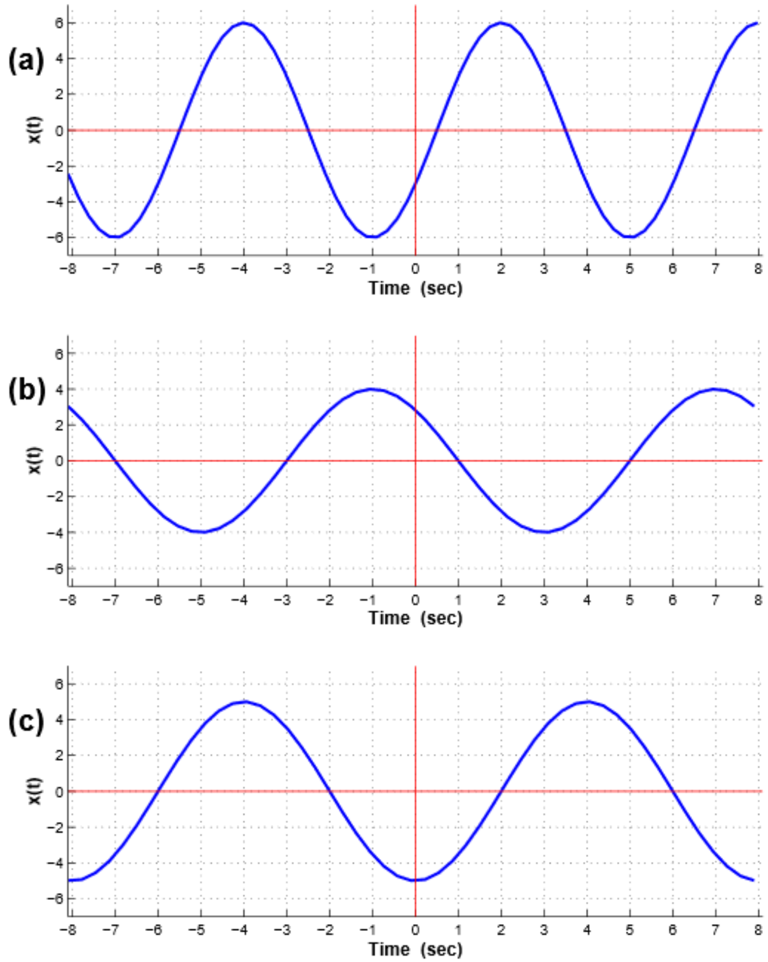
$= \cos(\pi t - 0.7854)$



# Q6

Several sinusoidal signals are plotted below. For each plot (a)–(c), determine the amplitude, phase (in radians) and frequency (in Hz). Write your answers in the following table:

| PLOT | (a) | (b) | (c) |
|---|---|---|---|
| AMPLITUDE | | | |
| PHASE (in radians) | | | |
| FREQUENCY (in Hz) | | | |

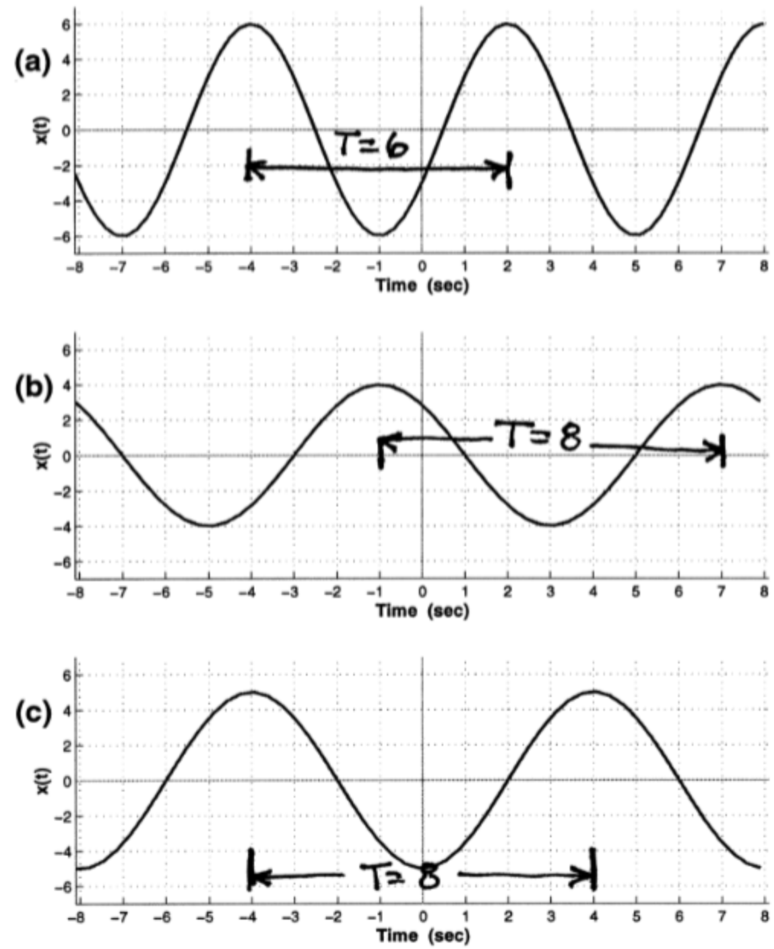


# Q6 Solution

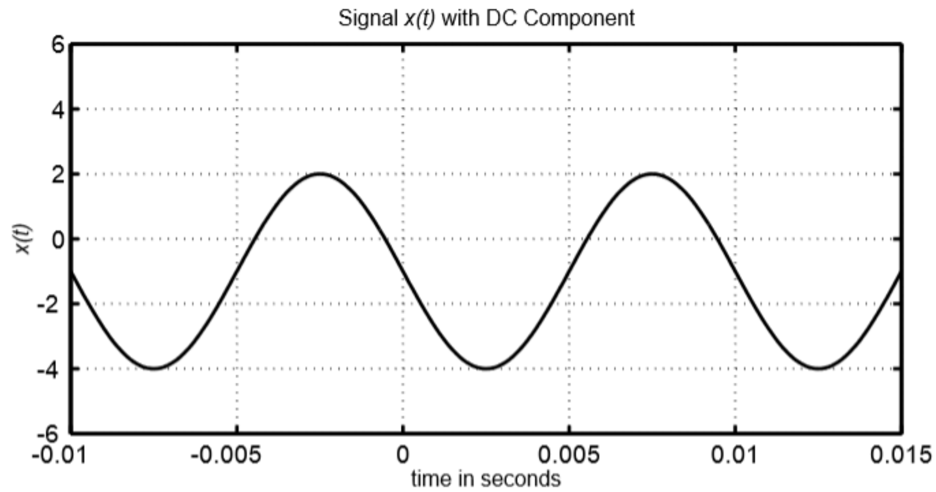
| PLOT | (a) | (b) | (c) |
|---|---|---|---|
| AMPLITUDE | 6 | 4 | 5 |
| PHASE (in radians) | $-2\pi/3$ $(4\pi/3)$ | $\pi/4$ $(-7\pi/4)$ | $\pi$ $(-\pi)$ |
| FREQUENCY (in Hz) | $\frac{1}{6}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |

EQUIVALENT BECAUSE OF PHASE AMBIGUITY

$\phi = -\omega t_m = -2\pi f t_m$



# Q7




The above signal $x(t)$ consists of a DC component plus a cosine signal. The terminology *DC component* means a component that is constant versus time.

(a) What is the frequency of the DC component? What is the frequency of the cosine component?

(b) Write an equation for the signal $x(t)$. You should be able to determine numerical values for all the amplitudes, frequencies, and phases in your equation by inspection of the above graph.

(c) Expand the equation obtained in the previous part into a sum of positive and negative frequency complex exponential signals.

(d) Then plot the two-sided spectrum of the signal $x(t)$. Show the complex amplitudes for each positive and negative frequency contained in $x(t)$.

# Q7 Solution

(A) The DC component has a frequency of zero, $f_{DC} = 0$. The cosine component has a frequency of $f = \frac{1}{T_0}$, where $T_0 \equiv$ period

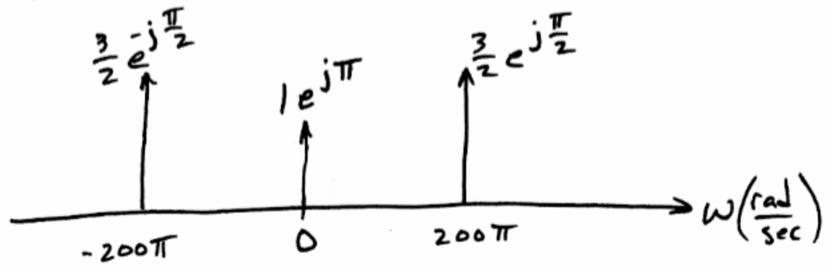
$$\Rightarrow f_{cosine} = \frac{1}{T_0} = \frac{1}{0.01} = 100 \text{ Hz}$$

(B) $$x(t) = -1 + 3\cos\left(200\pi t + \frac{\pi}{2}\right)$$

(C) $$x(t) = -1 + 3\left(\frac{e^{j\frac{\pi}{2}}e^{j200\pi t} + e^{-j\frac{\pi}{2}}e^{-j200\pi t}}{2}\right)$$

$$= -1 + \frac{3}{2}e^{j\frac{\pi}{2}}e^{j200\pi t} + \frac{3}{2}e^{-j\frac{\pi}{2}}e^{-j200\pi t}$$

(D)



# Q8

A periodic signal, $x(t)$, is given by

$$x(t) = 1 + 3\cos(300\pi t) + 2\sin(500\pi t - \pi/4)$$

(a) What is the period of $x(t)$?

(b) Find the Fourier series coefficients of $x(t)$.

# Q8 Solution

A periodic signal, $x(t)$, is given by

$$x(t) = 1 + 3\cos(300\pi t) + 2\sin(500\pi t - \pi/4)$$

(a) What is the period of $x(t)$?

The frequency of the first cosine in $x(t)$ is 150 Hz, and the frequency of the second is 250 Hz. Therefore, the fundamental frequency (the greatest common divisor) is $f_s = 50$ Hz. Thus, the period is

$$T = 1/f_s = 1/50$$

(b) Find the Fourier series coefficients of $x(t)$.

Use Euler's formula, $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$, to express the cosines in terms of complex exponentials

$$x(t) = 1 + \frac{3}{2}e^{j300\pi t} + \frac{3}{2}e^{-j300\pi t} + e^{j(500\pi t - 3\pi/4)} + e^{-j(500\pi t - 3\pi/4)}$$

so we have

$$a_0 = 1$$

$$a_3 = a_{-3} = 3/2$$

$$a_5 = a_{-5}^{*} = e^{-j3\pi/4}$$

# Q9

A signal $x(t)$ is periodic with period $T_0 = 8$. Therefore it can be represented as a Fourier series of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/8)kt}.$$

It is known that the Fourier series coefficients for this representation of a particular signal $x(t)$ are given by the integral

$$a_k = \frac{1}{8}\int_{-4}^{0} (4+t)e^{-j(2\pi/8)kt}\,dt. \tag{1}$$

(a) In the expression for $a_k$ in Equation (1) above, the integral and its limits define the signal $x(t)$. Determine an equation for $x(t)$ that is valid over one period.

(b) Using your result from part (a), draw a plot of $x(t)$ over the range $-10 \le t \le 10$ seconds. Label it carefully.

(c) Determine $a_0$, the DC value of $x(t)$.

# Q9 Solution

$T_0 = 8 \text{ (sec)} \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\left(\frac{2\pi}{8}\right)kt} \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8}$

Fourier coefficients: $\quad a_k = \frac{1}{8}\int_{-4}^{0}(4+t)\, e^{-j\frac{2\pi}{8}kt}\, dt$

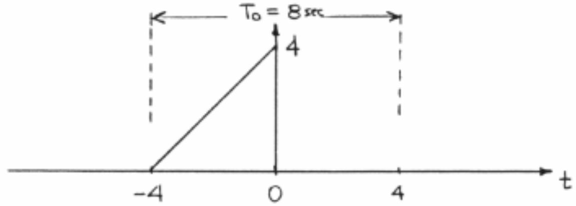
(a) The Fourier coefficients are given in general by:

$$a_k = \frac{1}{T_0}\int_{0}^{T_0} x(t)\, e^{-j\left(\frac{2\pi}{T_0}k\right)t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt$$

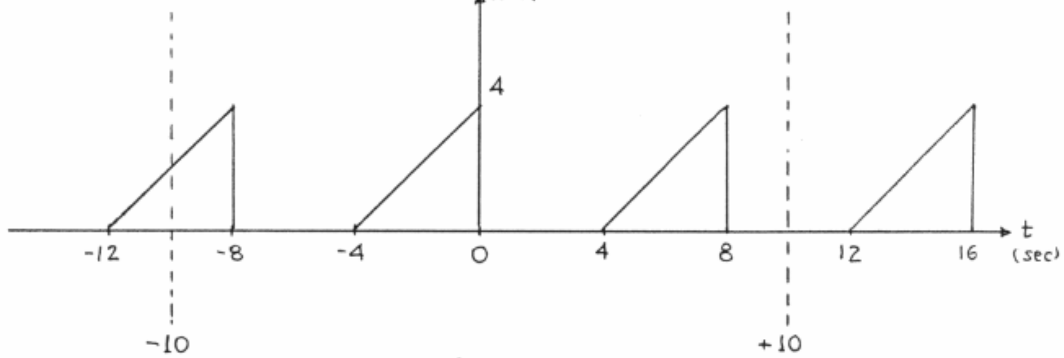
(any interval of length $T_0$)

From the given $a_k$ it can be observed that: $\quad (T_0 = 8)$

$$x(t) = \begin{cases} 4+t & -4 \leq t \leq 0 \\ 0 & 0 \leq t \leq 4 \end{cases} \qquad t \text{ in sec}$$




(b)




(c) $a_0 = \frac{1}{8}\int_{-4}^{4} x(t)\, dt = \frac{1}{8}\int_{-4}^{0}(t+4)\, dt = \frac{1}{8}\left(\frac{t^2}{2}+4t\right)\Big|_{-4}^{0} = 1$

# Q10

Consider the signal

$$x(t) = 8[\cos(1000\pi t)]^3.$$

(a) Using the inverse Euler relation for the sine function, express $x(t)$ as a sum of complex exponential signals with positive and negative frequencies.

(b) Use your result in part (a) to express $x(t)$ in the form $x(t) = A_1 \cos(\omega_0 t) + A_3 \cos(3\omega_0 t)$.

(c) Determine the period $T_0$ of $x(t)$ and sketch its waveform over the interval $-T_0 \le t \le 2T_0$. Carefully label the graph.

(d) Plot the spectrum of $x(t)$.

# Q10 Solution

$$x(t) = 8[\cos(1000\pi t)]^3 \quad \text{let } 1000\pi = \omega$$

(a)

$$x(t) = 8\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]^3 = \left(e^{j\omega t} + e^{-j\omega t}\right)^3$$

$$= \left(e^{j\omega t} + e^{-j\omega t}\right)\left(e^{j2\omega t} + e^{0} + e^{0} + e^{-j2\omega t}\right)$$

$$= e^{j3\omega t} + 2e^{j\omega t} + e^{j\omega t} + e^{j\omega t} + 2e^{-j\omega t} + e^{-j3\omega t}$$

$$= e^{j3\omega t} + 3e^{j\omega t} + 3e^{-j\omega t} + e^{-j3\omega t}$$

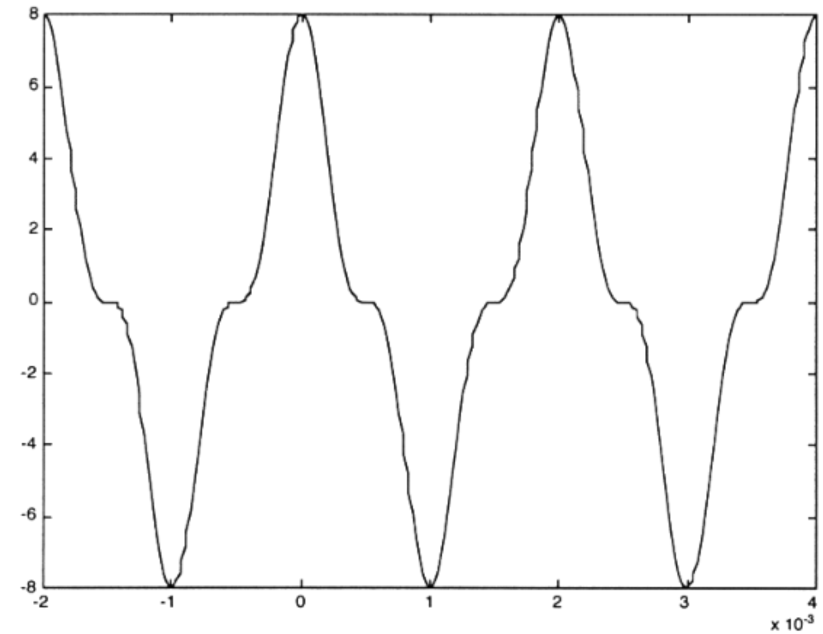
$$= e^{j3000\pi t} + 3e^{j1000\pi t} + 3e^{-j1000\pi t} + e^{-j3000\pi t}$$

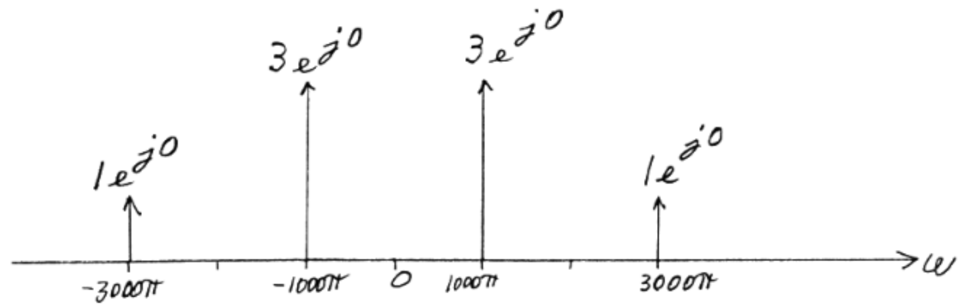
(b) $x(t) = 6\cos(1000\pi t) + 2\cos(3000\pi t)$

# Q10 Solution

(c)

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1000\pi} = 2\,ms$$



# Q11

A linear-FM "chirp" signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. We can define the *instantaneous frequency* of the chirp as the derivative of the "angle" of the sinusoid:

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) \tag{1}$$

where the cosine function operates on a time-varying angle argument

$$\psi(t) = \alpha t^2 + \beta t + \phi$$

The derivative of the angle argument $\psi(t)$ is the *instantaneous frequency,* which is also the audible frequency heard from the chirp. (The instantaneous frequency is the frequency heard by the human ear when the chirp rate is relatively slow. There are cases of FM where the audible signal is quite different, but these happen when the chirp rate is very high.)

$$\omega_i(t) = \frac{d}{dt}\psi(t) \qquad \text{radians/sec} \tag{2}$$

(a) For the "chirp" signal

$$x(t) = \Re e\left\{e^{j2\pi(-75t^2 + 900t + 33)}\right\}$$

derive a formula for the *instantaneous* frequency versus time.

(b) For the signal in part (b), make a plot of the *instantaneous* frequency (in Hz) versus time over the range $0 \le t \le 2$ sec.

# Q11 Solution

(a)

LINEAR CHIRP

$$x(t) = \mathrm{Re}\left\{ e^{j2\pi(-75t^2 + 900t + 33)} \right\}$$

$$= \cos\left[2\pi(-75t^2 + 900t + 33)\ \text{rad}\right]$$

$$= \cos\left[\text{phase}(t)\ \text{rad}\right]$$

SIGNAL

Phase of Signal

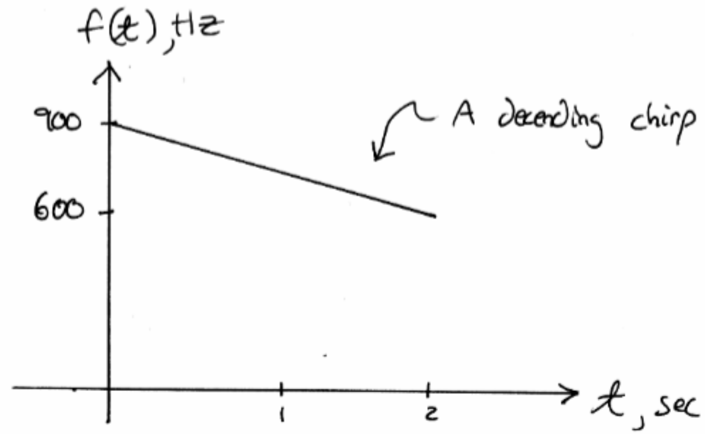
$$\frac{\partial\, \text{phase}(t)[\text{rad}]}{\partial t} = \omega(t)\left[\frac{\text{rad}}{\text{sec}}\right] = 2\pi(-150)t + 2\pi\, 900 + 0 = 2\pi f(t)$$

$$f(t) = -150t + 900 \quad [\text{Hz}]$$

(b)

$$f(t=0) = 900\ \text{Hz}$$

$$f(t=2\ \text{sec}) = -150(2) + 900 = 600\ \text{Hz}$$



# Q12

A periodic signal $x(t)$ with a period $T_0 = 4$ is described *over one period,* $0 \le t \le 4$, by the equation

$$x(t) = \begin{cases} 2 & 0 \le t \le 2 \\ 0 & 2 < t \le 4 \end{cases}$$

(a) Sketch the periodic function $x(t)$ for $-4 < t < 8$.

(b) Determine the D.C. coefficient of the Fourier Series, $a_0$.

(c) Use the Fourier $\boxed{\textit{analysis}}$ integral (for $k \neq 0$)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

to find the <u>first</u> Fourier series coefficient, $a_1$. Note: $\omega_0 = 2\pi/T_0$.

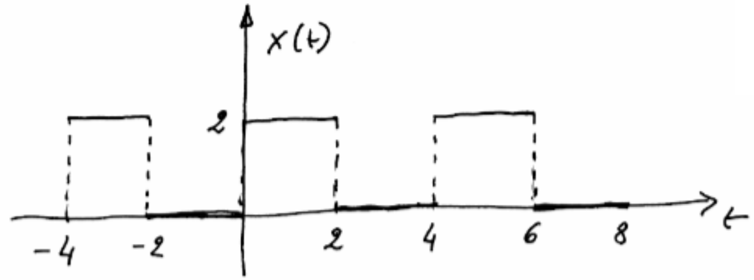
(d) Does the value of $a_1$ change if we add a constant value of one to $x(t)$, i.e., if we replace $x(t)$ with

$$x(t) = \begin{cases} 3 & 0 \le t \le 2 \\ 1 & 2 < t \le 4 \end{cases}$$

Explain why or why not. (Note: You should not have to evaluate $a_1$ explicitly to answer this question.)

# Q12 Solution

a)




b) $$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{4}\int_0^2 2\,dt = 1$$

c) $$a_1 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 t}\,dt = \frac{1}{4}\int_0^2 2e^{-j\omega_0 t}\,dt =$$

$$= \frac{1}{2}\left[\frac{e^{-j\omega_0 t}}{-j\omega_0}\right]_0^2 = \frac{j}{2\omega_0}\left(e^{-j2\omega_0} - 1\right) =$$

$$= \frac{j}{\pi}\left(e^{-j\pi} - 1\right) = -\frac{2j}{\pi}$$

(d) If $x(t)$ has a Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k\neq 0} a_k e^{jk\omega_0 t}$$

then the new signal

$$\hat{x}(t) = x(t) + 1$$

can be written as $\hat{x}(t) = (1+a_0) + \sum_{k\neq 0} a_k e^{jk\omega_0 t}$

# Q12 Solution

If we call the Fourier Series coefficients for $\hat{x}(t)$ $\hat{a}_k$, then

$$\hat{a}_0 = 1 + a_0$$
$$\hat{a}_k = a_k \quad \text{for } k \neq 0$$

d) Note that $\hat{x}(t) = x(t) + 1$. So:

$$\hat{a}_1 = \frac{1}{T_0}\int_0^{T_0} [x(t)+1]\, e^{-j\omega_0 t}$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\omega_0 t}\, dt + \frac{1}{T_0}\int_0^{T_0} e^{-j\omega_0 t}\, dt$$

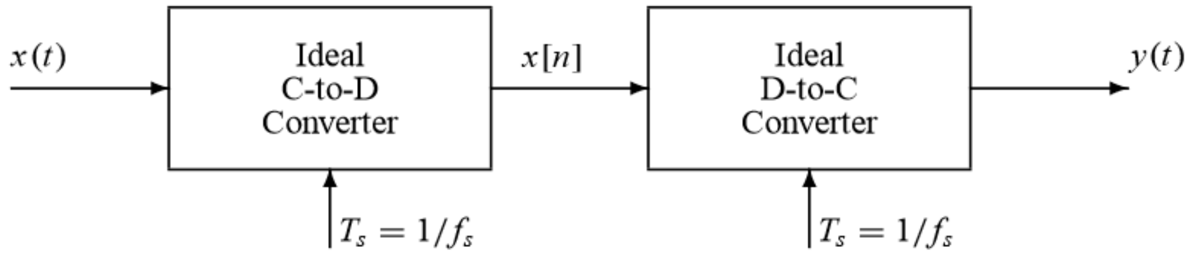
But: $\displaystyle\int_0^{T_0} e^{-j\omega_0 t}\, dt = \left[\frac{e^{-j\omega_0 t}}{-j\omega_0}\right]_0^{T_0} =$

$$= \frac{1}{-j\omega_0}\left(e^{-j\omega_0 T_0} - 1\right) = \frac{1}{-j\omega_0}\left(e^{-j2\pi} - 1\right) = 0$$

Therefore: $\displaystyle\hat{a}_1 = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\omega_0 t}\, dt = a_1$

# Q13

Consider the following system.




Suppose that the output of the C-to-D converter is

$$x[n] = 5 + 8\cos(0.4\pi n) + 4\cos(0.8\pi n + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 2000$ samples/second. Determine the output $y(t)$ of the ideal D-to-C converter.

# Q13 Solution

$$x[n] = 5 + 8\cos(0.4\pi n) + 4\cos(0.8\pi n + \pi/3)$$

$$x[n] \longrightarrow \boxed{D/C} \longrightarrow y(t)$$

$F_s = 2000$

For discrete to continuous, we replace "n" with $F_s t$

$$y(t) = x[n]\Big|_{n = F_s t}$$

$$= 5 + 8\cos(0.4\pi(2000)t) + 4\cos(0.8\pi(2000)t + \pi/3)$$

$$= 5 + 8\cos(2\pi(400)t) + 4\cos(2\pi(800)t + \pi/3)$$

# Q14

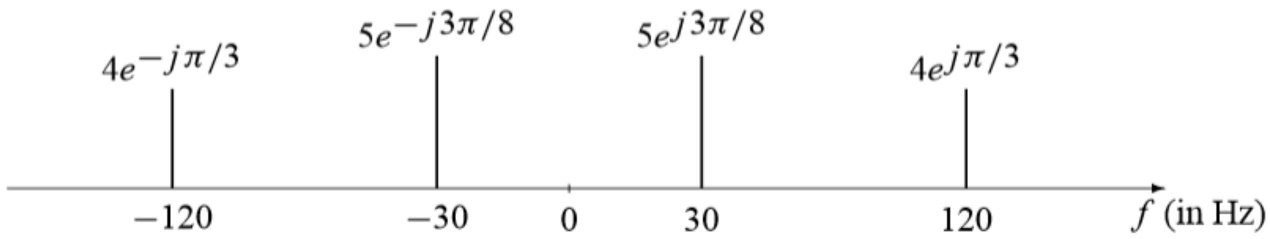
Again consider the ideal C-to-D converter and ideal D-to-C converter shown in previous problem.

(a) Suppose that a discrete-time signal $x[n]$ is given by the formula

$$x[n] = 4\cos(0.125\pi n + \pi/8)$$

If the sampling rate of the C-to-D converter is $f_s = 2000$ samples/second, many *different* continuous-time signals $x(t) = x_\ell(t)$ could have been inputs to the above system. Determine two such inputs with frequency less than 2000 Hz; i.e., find $x_1(t)$ and $x_2(t)$ such that $x[n] = x_1(nT_s) = x_2(nT_s)$ if $T_s = 1/2000$ secs.

(b) Now if the input $x(t)$ is given by the two-sided spectrum representation shown below,




Determine the spectrum for $x[n]$ when $f_s = 120$ samples/sec. Make a $\boxed{\text{plot}}$ for your answer, but label the frequency, amplitude and phase of each spectral component.

# Q14 Solution

Since $x[n]$ is given in sinusoids form, we suggest $x_1(t) = A_1\cos(2\pi f_1 t + \varphi_1)$ in which the parameters $A_1$, $f_1$, and $\varphi_1$ have to be found.

we use $x[n] = x_1(nT_s)$ where $T_s = \frac{1}{2000}$

$$\rightarrow 4\cos(0.125\pi n + \tfrac{\pi}{8}) = A_1\cos(2\pi f_1 \tfrac{n}{2000} + \varphi_1)$$

$$\Rightarrow A_1 = 4 \quad , \quad \varphi_1 = \frac{\pi}{8} \quad , \quad f_1 = 125 \text{ HZ}$$

Therefore, $x_1(t) = 4\cos(2\pi(125)t + \frac{\pi}{8})$

To find $x_2(t)$ (that would give the same $x[n]$), we use the fact that adding $2\pi k$ $(k = \pm 1, \pm 2, \dots)$ to $\hat{\omega}$ (the frequency of $x[n]$) does not change anything. i.e., $x[n] = 4\cos((0.125\pi + 2k\pi)n + \frac{\pi}{8})$

If we start with $x_2(t) = A_2\cos(2\pi f_2 t + \varphi_2)$, then $x_2(t)$ can be found from $x_2(nT_s) = x[n]$

# Q14 Solution

Following the same approach we had for $x_1(t)$, we obtain:

$$A_2 = 4 \quad , \quad \varphi_2 = \frac{\pi}{8} \quad ,$$

$$\frac{2\pi f_2}{2000} = 0.125\pi + 2K\pi \qquad K = \pm 1, \pm 2, \dots$$

$$\rightarrow f_2 = 125 + 2000K \qquad K = \pm 1, \pm 2, \dots$$

Since we require $f_2 < 2000$, thus we choose

$K = -1 \rightarrow f_2 = -1875$

$$\rightarrow x_2(t) = 4\cos\left(-2\pi(1875)t + \frac{\pi}{8}\right)$$

$$= 4\cos\left(2\pi(1875)t - \frac{\pi}{8}\right)$$

because $\cos(-\theta) = \cos(\theta)$ for any $\theta$

# Q14 Solution

(b) First we find $x(t)$ from the spectrum:

$$x(t) = 10\cos(2\pi(30)t + \frac{3\pi}{8}) + 8\cos(2\pi(120)t + \frac{\pi}{3})$$

Now, we obtain $x[n]$. Since $f_s = 120$ Samples/sec we expect that there is an aliasing term introduced by the second cosine term in $x(t)$.

$x[n] = x(nT_s)$ where $T_s = \frac{1}{120}$

$$x[n] = 10\cos(\frac{\pi}{2}n + \frac{3\pi}{8}) + 8\cos(2\pi n + \frac{\pi}{3})$$

Thus $x[n]$ has two frequency components:

$\hat{\omega}_1 = \frac{\pi}{2}$ , $\hat{\omega}_2 = 2\pi$

To plot the spectrum of $x[n]$, we treat it similar to the plot of the spectrum for continuous-time signals. But we need to keep in mind two things:

(1) We plot the spectrum of any arbitrary $x[n]$ in the interval $-\pi \leq \hat{\omega} \leq \pi$.

(2) For those frequency components that are not in the interval $-\pi \leq \omega \leq \pi$, we subtract (or add) multiples of $(2\pi)$ such

# Q14 Solution

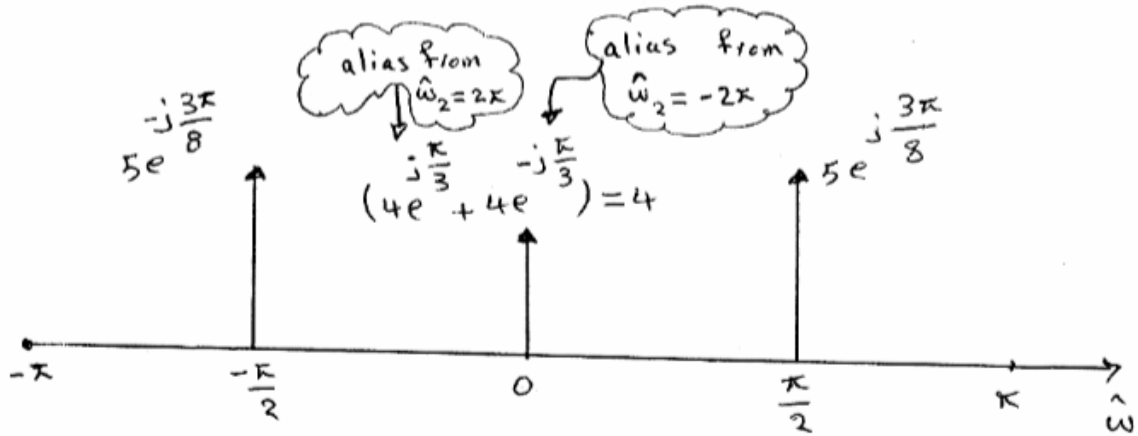
that the new frequency lies in the interval

$-\pi \le \hat{\omega} \le \pi$.

Note that step (2) in the above is required for all aliasing terms.

Now, since $\hat{\omega}_1 = \frac{\pi}{2}$, step (2) is not required.

But for $\hat{\omega}_2 = 2\pi$, we follow step (2). This gives a new $\hat{\omega}_2 = 2\pi - 2\pi = 0$




Note to the DC term introduced by aliasing frequecies.

# Q15

The "spectrum" diagram gives the frequency content of a signal.

(a) Draw a sketch of the spectrum of $x(t)$ which is "cosine-times-sine"

$$x(t) = \cos(50\pi t)\ \sin(700\pi t)$$

Label the frequencies and complex amplitudes of each component.

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

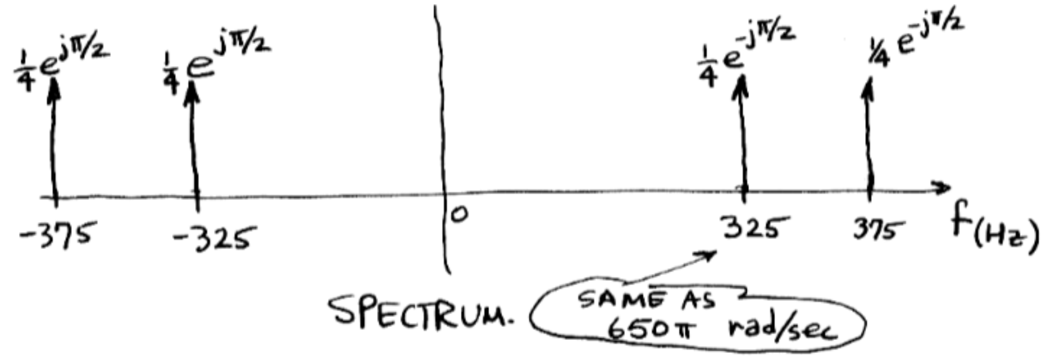
# Q15 Solution

$$x(t) = \cos(50\pi t)\sin(700\pi t)$$

(a) $$x(t) = \left(\frac{1}{2}e^{j50\pi t} + \frac{1}{2}e^{-j50\pi t}\right)\left(\frac{1}{2j}e^{j700\pi t} - \frac{1}{2j}e^{-j700\pi t}\right)$$

$$= \frac{1}{4j}e^{j750\pi t} + \frac{1}{4j}e^{j650\pi t} - \frac{1}{4j}e^{-j650\pi t} - \frac{1}{4j}e^{-j750\pi t}$$

SAME AS $\frac{1}{4}e^{-j\pi/2}$ SAME AS $\frac{1}{4}e^{+j\pi/2}$




SPECTRUM.

(b) Sampling Thm says sample at a rate greater than $\boxed{\text{two}}\ \boxed{\text{times}}$ the highest freq.

HIGHEST FREQ = 375 Hz

$\Rightarrow f_s \geq 750$ Hz.