Digital Signal Processing

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Q1

Use the following trigonometric identity to derive an expression for $\cos 8\theta$ in terms of $\cos 9\theta$, $\cos 7\theta$, and $\cos\theta$.

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

2

A1

$$\cos 9\theta = \cos(8\theta + \theta) = \cos 8\theta \cos\theta - \sin 8\theta \sin\theta$$
$$\cos 7\theta = \cos(8\theta - \theta) = \cos 8\theta \cos\theta + \sin 8\theta \sin\theta$$

$$\cos 9\theta + \cos 7\theta = 2\cos\theta \cos 8\theta$$
$$\Rightarrow \cos 8\theta = \frac{\cos 9\theta + \cos 7\theta}{2\cos\theta}$$

3

Q2

Plot the graph of the following funtion:

$$s(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ \frac{1}{3}t(4-2t) & \frac{1}{2} \le t \le 2 \\ 0 & elsewhere \end{cases}$$

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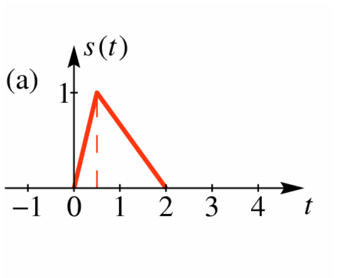
A2



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Q3

Considering the funtion given in question 2, derive an equation for $x_1(t) = s(t-2)$ and plot the graph of the funtion.

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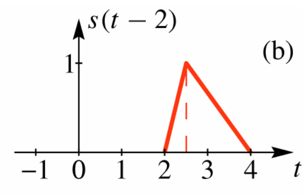
A3

$$x_1(t) = s(t-2) = \begin{cases} 2(t-2) & 2 \le t \le 2\frac{1}{2} \\ \frac{1}{3}(8-2t) & 2\frac{1}{2} \le t \le 4 \\ 0 & elsewhere \end{cases}$$



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Q4

Considering the funtion given in question 2, derive an equation for $x_2(t) = s(t+1)$ and plot the graph of the funtion.

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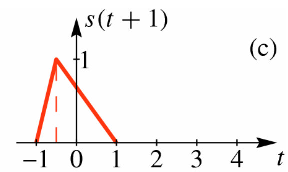
A4

$$x_2(t) = s(t+1) = \begin{cases} 2(t+2) & -1 \le t \le -\frac{1}{2} \\ \frac{1}{3}(2-2t) & -\frac{1}{2} \le t \le 1 \\ 0 & elsewhere \end{cases}$$



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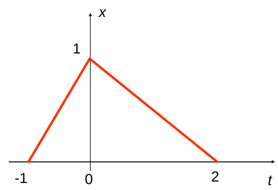
Q5

Derive an equation for the following graph of the funtion.



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A5

$$x(t) = \begin{cases} (t+1) & -1 \le t \le 0 \\ (1-\frac{1}{2}t) & 0 \le t \le 2 \\ 0 & elsewhere \end{cases}$$

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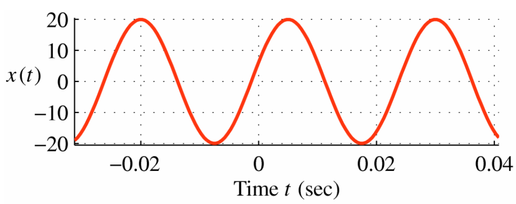
Q6

In the following waveform, it is possible to measure both a positive and a negative value of t_1 and then calculate the corresponding phase shifts.

Which phase shift is within the range $-\pi < \varphi \le \pi$? Verify that the two phase shifts differ by 2π.



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A6

Positive t_1 is $t_1 = 0.005$ sec

$$\varphi = -\omega_0 t_1 = -2\pi(40)(0.005) = -2\pi(0.2)$$

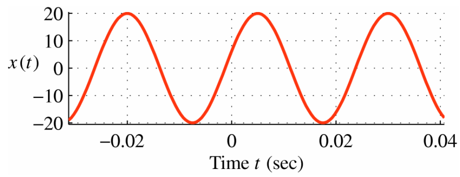
Negative t_1 is $t_1 = -0.02$ sec

$$\varphi = -\omega_0 t_1 = -2\pi(40)(-0.02) = 2\pi(0.8)$$

Difference = $2\pi(0.8) - (-2\pi(0.2)) = 2\pi$

Q7

For the following signal, $x(t)=20\cos(2\pi(40)t - 0.4\pi)$, find G and t_1 so that the signal $y(t)=Gx(t - t_1)$ is equal to $5\cos(2\pi(40)t)$; i.e., obtain an expression for $y(t)= 5\cos(2\pi(40)t)$ in terms of $x(t)$.



A7

$$y(t) \stackrel{?}{=} G\,x(t-t_1)$$

$$5\cos(2\pi(40)t) \stackrel{?}{=} G(20)\cos(\underbrace{2\pi(40)(t-t_1)-0.4\pi}_{2\pi(40)t - 80\pi t_1 - 0.4\pi})$$

$\Rightarrow 5 = 20G$

$\boxed{\therefore G = \frac{5}{20} = \frac{1}{4}}$

NEED THIS PART TO BE ZERO

$\Rightarrow -80\pi t_1 - 0.4\pi = 0$

$\therefore t_1 = \frac{0.4\pi}{-80\pi} = -\frac{1}{200}$ sec $= -0.005$

Q8

Demonstrate that expanding the real part of *exp(j(α+β)) = exp(jα)×exp(jβ)* will lead to the following identity.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Also show that the following identity is obtained from the imaginary part.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

A8

$$e^{j(\alpha+\beta)} = e^{j\alpha}e^{j\beta}$$

$$= (\cos\alpha + j\sin\alpha)(\cos\beta + j\sin\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + j(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

EQUALS $\cos(\alpha+\beta) + j\sin(\alpha+\beta)$

EQUATE REAL PARTS:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

EQUATE IMAGINARY PARTS:

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

Q9

Show that the following representation can be derived for the real sine signal:

$$A\sin(\omega_0 t + \phi) = \tfrac{1}{2}Xe^{-j\pi/2}e^{j\omega_0 t} + \tfrac{1}{2}X^*e^{j\pi/2}e^{-j\omega_0 t}$$

where $X = Ae^{j\phi}$. In this case, the interpretation is that the sine signal is also composed of two complex exponentials with the same positive and negative frequencies, but the complex coefficients multiplying the terms are different from those of the cosine signal. Specifically, the sine signal requires additional phase shifts of $\pm\pi/2$ applied to the complex amplitude X and X^*, respectively.

A9

$$\tfrac{1}{2} A e^{j\varphi} e^{-j\pi/2} e^{j\omega_0 t} + \tfrac{1}{2} A e^{-j\varphi} e^{j\pi/2} e^{-j\omega_0 t}$$

CONJUGATE

$$= \tfrac{1}{2} A \left(e^{j(\omega_0 t + \varphi - \pi/2)} + e^{-j(\omega_0 t + \varphi - \pi/2)} \right)$$

$$= A \cos(\omega_0 t + \varphi - \pi/2)$$

USE $\tfrac{1}{2} e^{j\theta} + \tfrac{1}{2} e^{-j\theta} = \cos\theta$

RECALL TRIG. IDENTITY: $\cos(\alpha - \pi/2) = \sin\alpha$

$$A \sin(\omega_0 t + \varphi)$$

Q10

Use $A_k\cos(\omega_0 t+\phi_k)=A_k\cos\phi_k \cos(\omega_0 t) - A_k\sin\phi_k \sin(\omega_0 t)$

to show that the sum

$1.7\cos(20\pi t - 70\pi/180)+ 1.9\cos(20\pi t + 200\pi/180)$

reduces to $A\cos(20\pi t+\phi)$, where

$A= 1.532$ and

$\phi = 2.475$ rads

A10

Recall the trig identity:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$1.7\cos(2\pi(10)t + 70\pi/180)$ expands into two terms:

$$1.7\cos\tfrac{70\pi}{180}\cos(2\pi(10)t) - 1.7\sin\tfrac{70\pi}{180}\sin(2\pi(10)t)$$

$1.9\cos(2\pi(10)t + 200\pi/180)$ expands to:

$$1.9\cos\tfrac{200\pi}{180}\cos(2\pi(10)t) - 1.9\sin\tfrac{200\pi}{180}\sin(2\pi(10)t)$$

ADD THESE

The sum is:

$$\underbrace{\left(1.7\cos\tfrac{70\pi}{180} + 1.9\cos\tfrac{200\pi}{180}\right)}_{\text{call this } P}\cos(2\pi(10)t) - \underbrace{\left(1.7\sin\tfrac{70\pi}{180} + 1.9\sin\tfrac{200\pi}{180}\right)}_{\text{call this } Q}\sin(2\pi(10)t)$$

Now we need another identity for:

$$P\cos(2\pi(10)t) - Q\sin(2\pi(10)t)$$

Here's one way to do it:

$$\sqrt{P^2+Q^2}\left(\frac{P}{\sqrt{P^2+Q^2}}\cos(2\pi(10)t) - \frac{Q}{\sqrt{P^2+Q^2}}\sin(2\pi(10)t)\right)$$

If we notice that P & Q are sides of a triangle with hypotenuse $\sqrt{P^2+Q^2}$, then we can define φ so that

$$\cos\varphi = \frac{P}{\sqrt{P^2+Q^2}} \quad \& \quad \sin\varphi = \frac{Q}{\sqrt{P^2+Q^2}}$$

Now the identity becomes:

$$\sqrt{P^2+Q^2}\left(\cos\varphi\cos(2\pi(10)t) - \sin\varphi\sin(2\pi(10)t)\right)$$

and we use the first identity to get

$$\sqrt{P^2+Q^2}\cos(2\pi(10)t + \varphi)$$

Now we can write out φ & $\sqrt{P^2+Q^2}$:

$$A = \sqrt{P^2+Q^2} = (P^2+Q^2)^{1/2}$$

Amplitude

$$= \left\{\left[1.7\cos\tfrac{70\pi}{180} + 1.9\cos\tfrac{200\pi}{180}\right]^2 + \left[1.7\sin\tfrac{70\pi}{180} + 1.9\sin\tfrac{200\pi}{180}\right]^2\right\}^{1/2}$$

and for the angle φ:

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{Q}{P}$$

$$\Rightarrow \varphi = \mathrm{ArcTan}\left\{\frac{1.7\sin\frac{70\pi}{180} + 1.9\sin\frac{200\pi}{180}}{1.7\cos\frac{70\pi}{180} + 1.9\cos\frac{200\pi}{180}}\right\}$$

Q11

Consider the two sinusoids,

$x_1(t) = 5\cos(2\pi(100)t + \pi/3)$

$x_2(t) = 4\cos(2\pi(100)t - \pi/4)$

Obtain the phasor representations of these two signals, add the phasors, plot the two phasors and their sum in the complex plane, and show that the sum of the two signals is

$x_3(t) = 5.536\cos(2\pi(100)t + 0.2747)$

In degrees the phase should be 15:74. Examine the plots in Fig. 2-16 to see whether you can identify the cosine waves

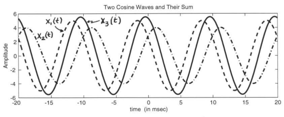
- $x1.t/$, $x2.t/$, and $x3.t/$ D $x1.t/$ C $x2.t/$.

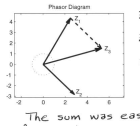
A11

$$x_1(t) = 5\cos(2\pi(100)t + \pi/3) \Rightarrow Z_1 = 5e^{j\pi/3}$$

Amp phase

$$x_2(t) = 4\cos(2\pi(100)t - \pi/4) \Rightarrow Z_2 = 4e^{-j\pi/4}$$





$$Z_1 = 5e^{j\pi/3} = 2.5 + j4.33$$

$$Z_2 = 4e^{-j\pi/4} = 2.828 - j2.828$$

$$Z_3 = Z_1 + Z_2$$

$$Z_3 = 5.328 + j1.502$$

The sum was easier to do in Cartesian form, but now we convert to polar form

$$Z_3 = 5.536e^{j0.275} = 5.536e^{j0.087\pi}$$

15.74°

Amp phase

$$x_3(t) = 5.536\cos(2\pi(100)t + 0.275)$$

keep the same frequency

Q12

Demonstrate that a complex exponential signal can also be a solution to the tuning-fork differential equation:

$d^2x/dt^2 = -(k/m)x(t)$

By substituting $z(t)$ and $z^*(t)$ into both sides of the differential equation, show that the equation is satised for all t by both of the signals

$z(t)= X e^{j\omega_0 t}$ and $z^*(t)= X^* e^{j\omega_0 t}$

Determine the value of ω_0 for which the differential equation is satised.

A12

FIRST DERIVATIVE:

$$\frac{d}{dt}\bar{x}(t) = j\omega_0 \mathbb{X} e^{j\omega_0 t} \qquad \frac{d}{dt}\bar{x}^*(t) = -j\omega_0 \mathbb{X}^* e^{-j\omega_0 t}$$

2ND DERIVATIVE:

$$\frac{d^2}{dt^2}\bar{x}(t) = \underbrace{(j\omega_0)^2}_{\text{EQUALS } -\omega_0^2} \mathbb{X} e^{j\omega_0 t} \qquad \frac{d^2}{dt^2}\bar{x}^*(t) = \underbrace{(-j\omega_0)^2}_{= -\omega_0^2} \mathbb{X}^* e^{-j\omega_0 t}$$

If $-\omega_0^2 = -\frac{k}{m}$, then the differential equation is satified $\quad \therefore \ \omega_0 = \sqrt{\frac{k}{m}}$

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