

Digital Signal Processing

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Digital Signal Processing

Lecture 22

Sampling and Reconstruction (Fourier View)

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LECTURE OBJECTIVES

- **Sampling Theorem** Revisited
 - GENERAL: in the FREQUENCY DOMAIN
 - Fourier transform of sampled signal
 - Reconstruction from samples
- Reading: Chap 12, Section 12-3
- Review of FT properties
 - Convolution \leftrightarrow multiplication
 - Frequency shifting
 - Review of AM

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

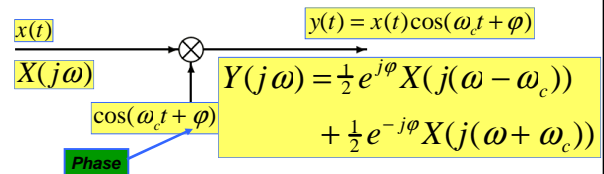
Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

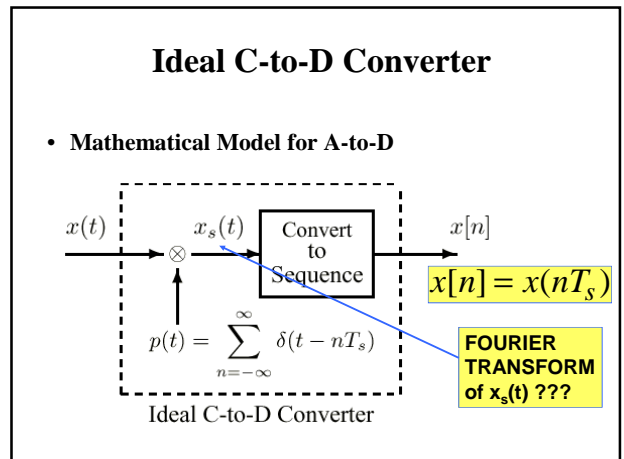
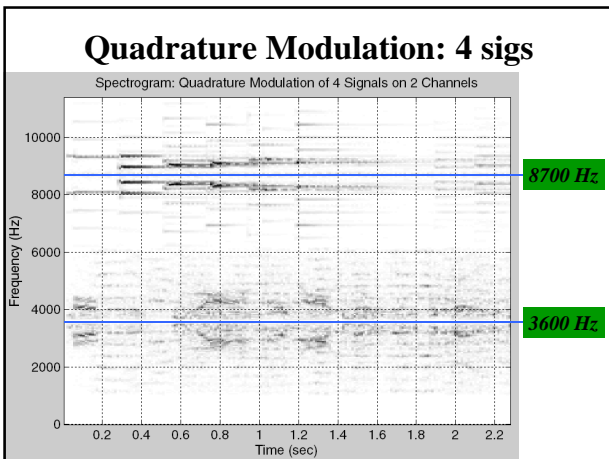
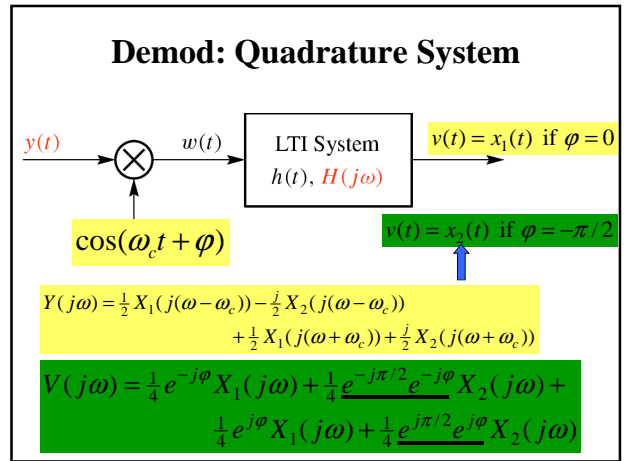
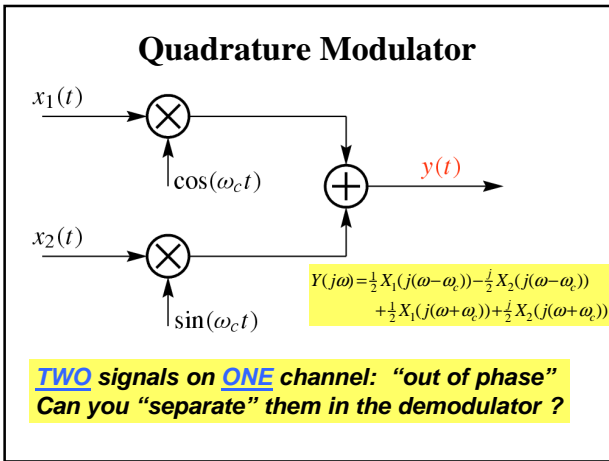
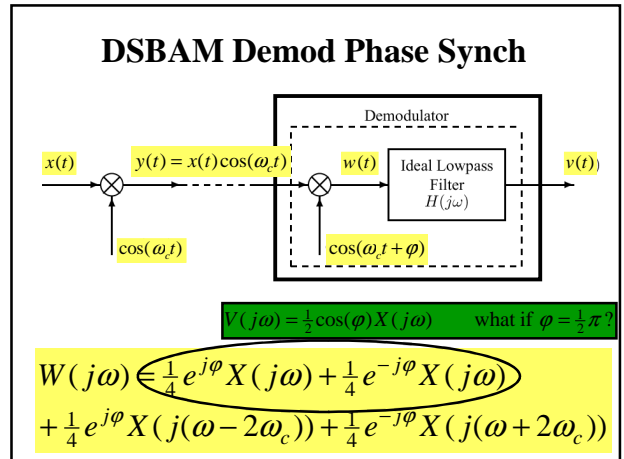
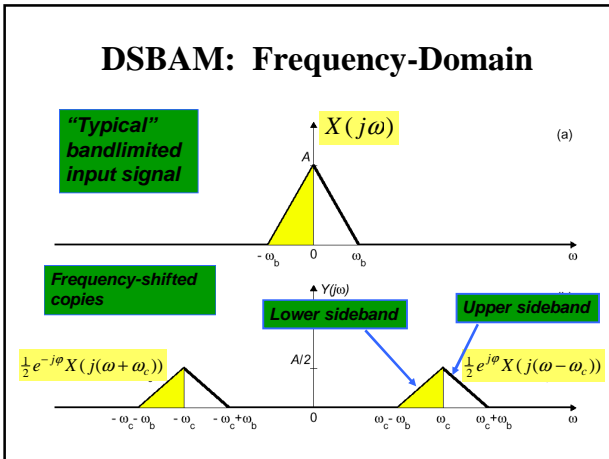
Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\left(\frac{\omega}{a}\right))$$

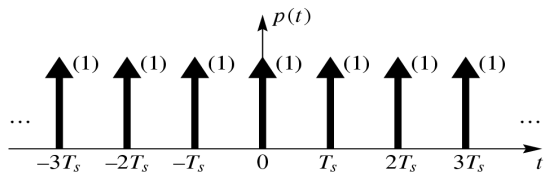
Amplitude Modulator



- $x(t)$ modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of $X(j\omega)$.



Periodic Impulse Train

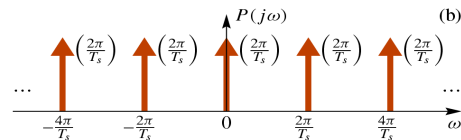
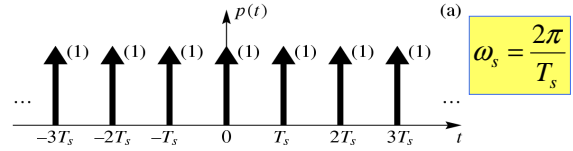


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t} \quad \omega_s = \frac{2\pi}{T_s}$$

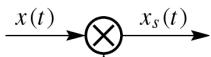
$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s} \quad \text{Fourier Series}$$

FT of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \leftrightarrow P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$



Impulse Train Sampling

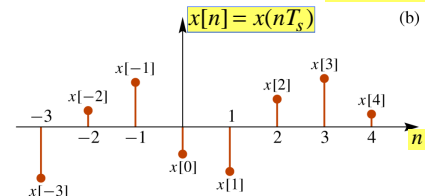
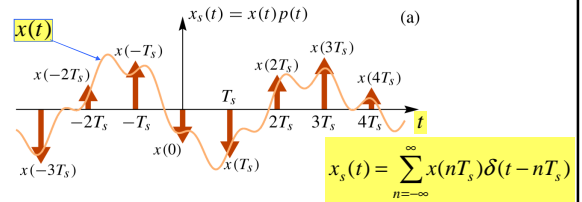


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

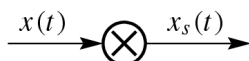
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

Illustration of Sampling



Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t} \quad \text{EXPECT FREQUENCY SHIFTING !!!}$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

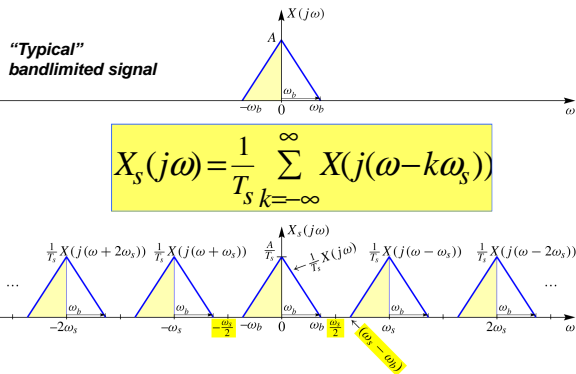
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

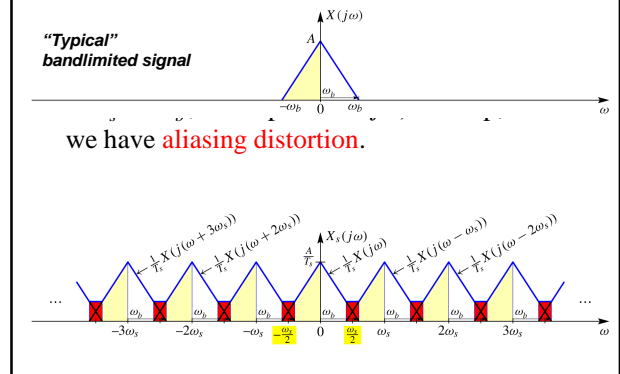
$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

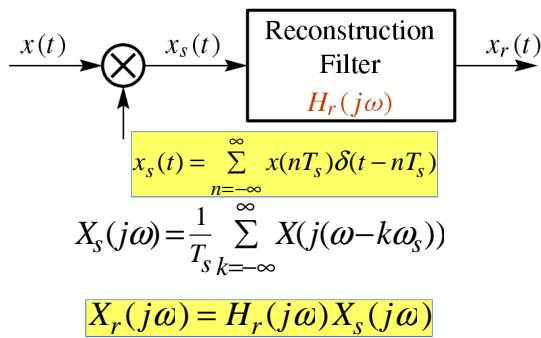
Frequency-Domain Representation of Sampling



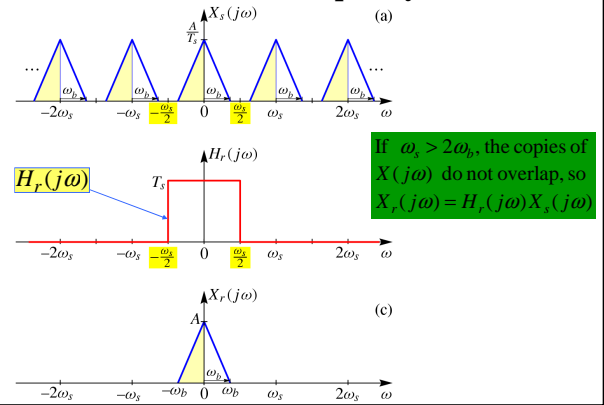
Aliasing Distortion



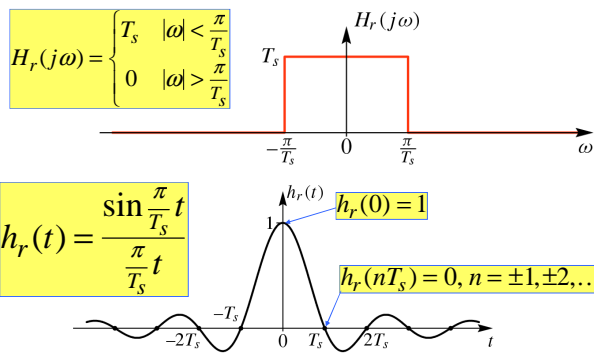
Reconstruction of x(t)



Reconstruction: Frequency-Domain



Ideal Reconstruction Filter



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

Ideal bandlimited interpolation formula

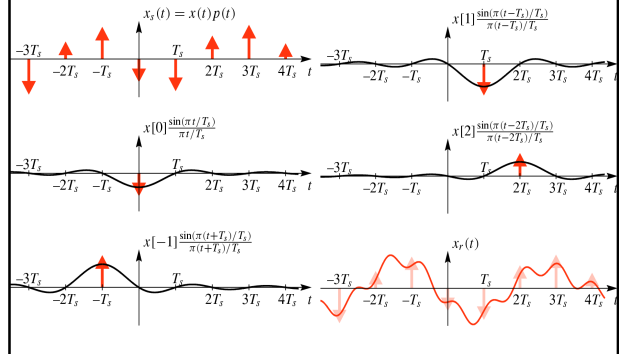
Shannon Sampling Theorem

- **“SINC” Interpolation** is the ideal
 - PERFECT RECONSTRUCTION
 - of BANDLIMITED SIGNALS

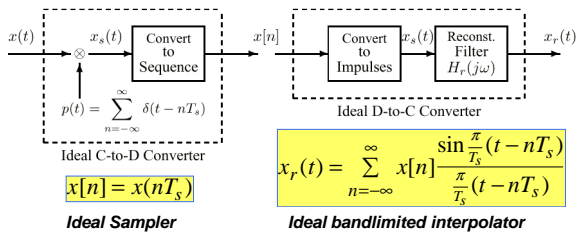
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \left[\frac{\pi}{T_s} (t - nT_s) \right]}{\frac{\pi}{T_s} (t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

$$X_S(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$