Digital Signal Processing

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Digital Signal Processing

Lecture 20

Fourier Transform Properties

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, Sects. 11-5 to 11-9
  - **Tables in Section 11-9**
- Other Reading:
  - Recitation: Chapter 11, Sects. 11-1 to 11-9
  - Next Lectures: Chapter 12 (Applications)

LECTURE OBJECTIVES

- The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis (Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$

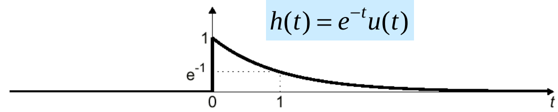
$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- Manipulate the **"Frequency Spectrum"**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the "**Building Blocks**" ?
    - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers or Multipliers: $x(t)p(t)$

Frequency Response

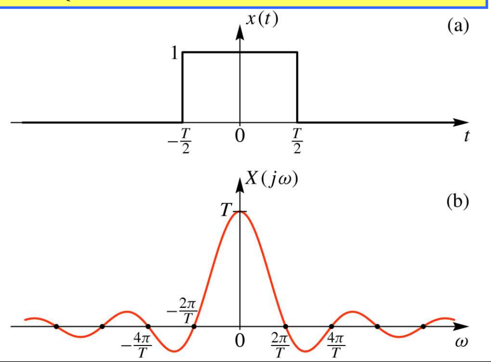
- Fourier Transform of $h(t)$ **is** the Frequency Response



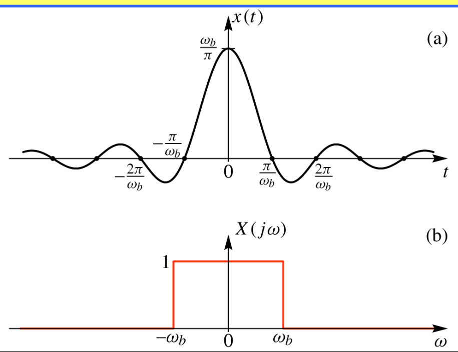
$h(t) = e^{-t}u(t)$

$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$



$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$t_0 = 0$

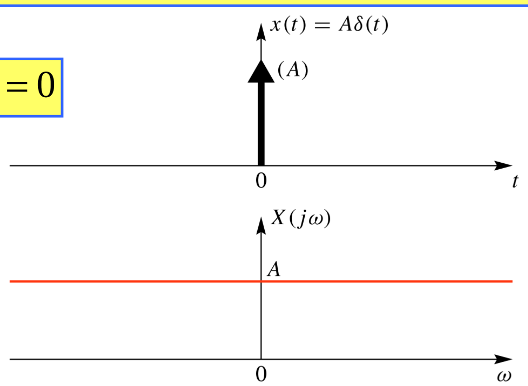


Table of Fourier Transforms

$$x(t) = e^{-t}u(t) \Leftrightarrow X(j\omega) = \frac{1}{1 + j\omega}$$

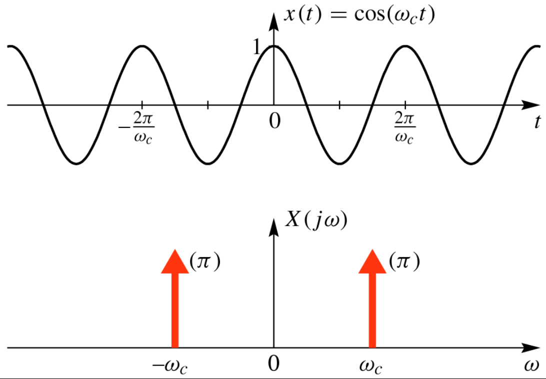
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_c t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_c)$$

$$x(t)=\cos(\omega_c t) \Leftrightarrow X(j\omega)=\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$$



Fourier Transform of a General Periodic Signal

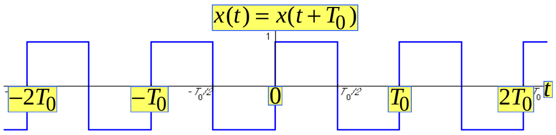
- If $x(t)$ is periodic with period T_0 ,

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k=\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-k\omega_0)$

$$X(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

Square Wave Signal



$$a_k=\frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt+\frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k=\left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2}-\left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0}=\frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform

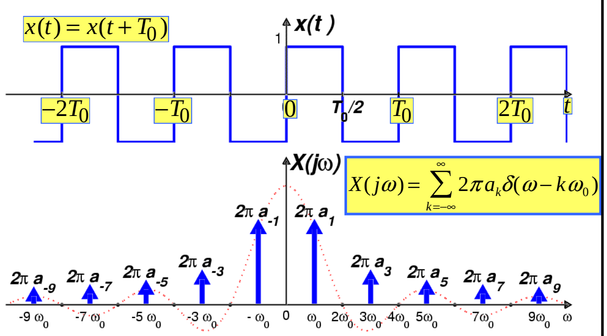


Table of Easy FT Properties

Linearity Property

$$ax_1(t)+bx_2(t) \Leftrightarrow aX_1(j\omega)+bX_2(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Scaling

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Scaling Property

$$x(at) \Leftrightarrow \tfrac{1}{|a|} X(j\tfrac{\omega}{a})$$

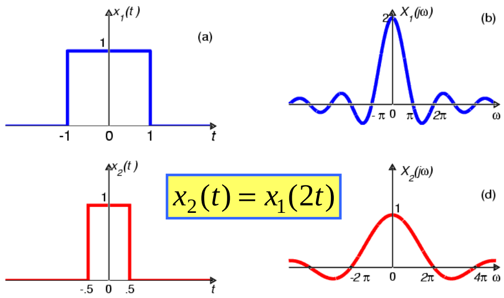
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt=\int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)}\tfrac{d\lambda}{|a|}$$

$$=\tfrac{1}{|a|} X(j\tfrac{\omega}{a})$$

$x(2t)$ shrinks; $\frac{1}{2}X(j\frac{\omega}{2})$ expands

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$



Uncertainty Principle

- Try to make $x(t)$ shorter
  - Then $X(j\omega)$ will get wider
  - Narrow pulses have wide bandwidth
- Try to make $X(j\omega)$ narrower
  - Then $x(t)$ will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

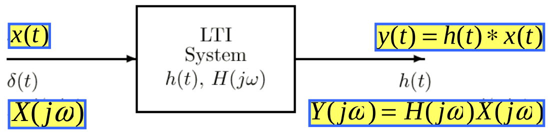
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

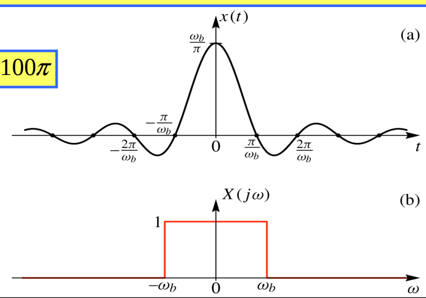
Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function
- Ideal LPF (Lowpass Filter)
  - $h(t)$ is a "sinc"
- **Output** is Bandlimited
  - Convolve "sincs"

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

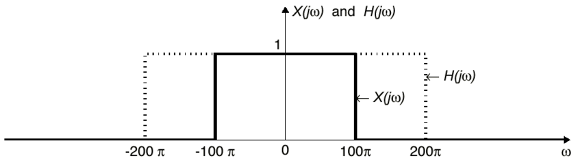
$\omega_b = 100\pi$



Convolution Example

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

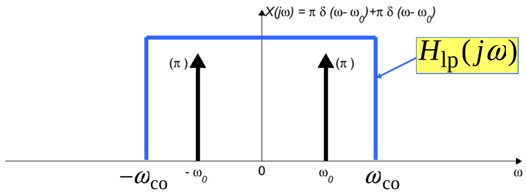
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega-\omega_0)+H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter



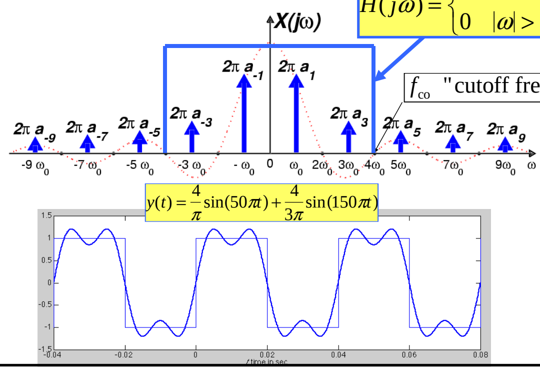
$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter

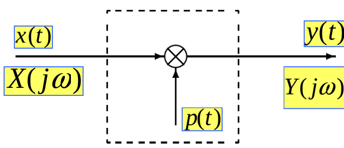
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

f_{co} "cutoff freq."



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

Signal Multiplier (Modulator)



$$y(t) = p(t)x(t)$$

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega)*P(j\omega)$$

- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

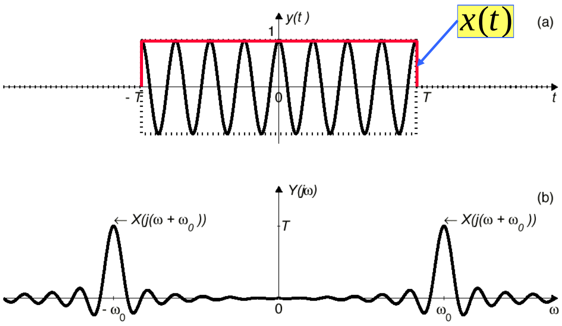
$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

$$= X(j(\omega-\omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0-7 < \omega < \omega_0+7 \\ 0 & elsewhere \end{cases}$$

$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$

$$Y(j\omega) = \tfrac{1}{2}X(j(\omega-\omega_0)) + \tfrac{1}{2}X(j(\omega+\omega_0))$$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$

$$y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$p(t) = \cos(\omega_0 t) \Leftrightarrow P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * [\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$$

$$Y(j\omega) = \tfrac{1}{2}X(j(\omega-\omega_0)) + \tfrac{1}{2}X(j(\omega+\omega_0))$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d}X(j\omega)$$

$$\int_{-\infty}^{\infty} x(t-t_d)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_d)}d\tau$$

$$= e^{-j\omega t_d}X(j\omega)$$

For example, $e^{-a(t-5)}u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a+j\omega}$

Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of "theorems" or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.

FT of Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)e^{-j\omega_0 t}dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\therefore \quad P(j\omega) = \left(\frac{2\pi}{T_0}\right)\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)$$

Convolution Example 2

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$$

$$y(t) = x(t) * h(t)$$