Digital Signal Processing

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Digital Signal Processing

Lecture 18

3-Domains for IIR

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, all
- Other Reading:
  - Recitation: Ch. 8, all
    - POLES & ZEROS
  - Next Lecture: Chapter 9

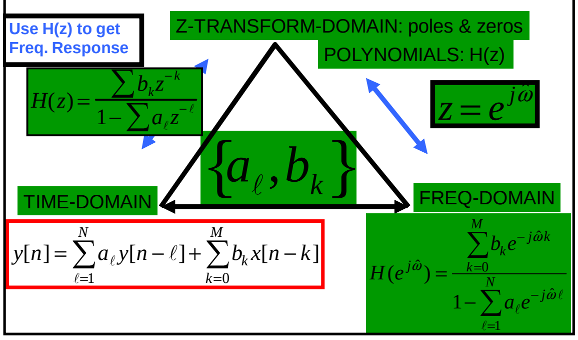
LECTURE OBJECTIVES

- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- $H(z)$ can have COMPLEX POLES & ZEROS
- THREE-DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

THREE DOMAINS



Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$		$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

$$(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make **BANDPASS FILTER**

$$\text{pole} = re^{j\theta}$$

$$r \to 1 ?$$

2nd ORDER EXAMPLE

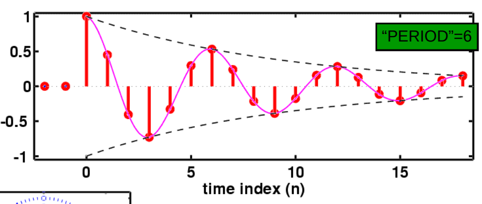
$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

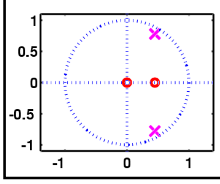
$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9e^{j\pi/3} z^{-1})(1 - 0.9e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$h[n]$: Decays & Oscillates





$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = A r^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = A \frac{\cos\varphi - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

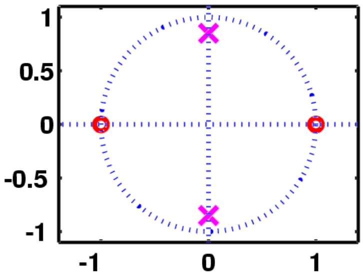
2nd ORDER EX: n-Domain

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

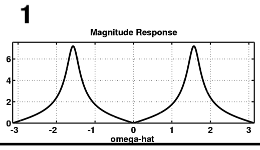
$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - 0.45x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```

Complex POLE-ZERO PLOT

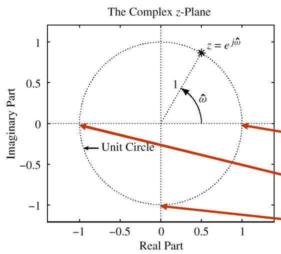


$$\frac{1 - z^{-2}}{1 + 0.7225 z^{-2}}$$



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



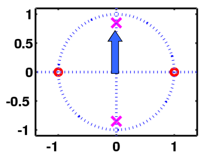
$$z = e^{j\hat{\omega}}$$

$$z = 1 \leftrightarrow \hat{\omega} = 0$$

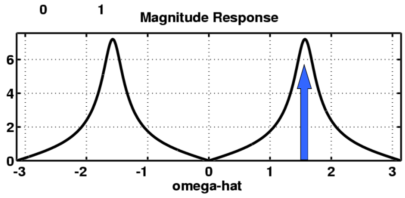
$$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$$

$$z = \pm j \leftrightarrow \hat{\omega} = \pm\tfrac{1}{2}\pi$$

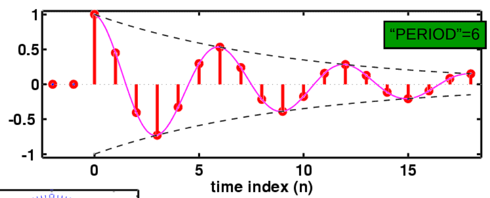
FREQUENCY RESPONSE from POLE-ZERO PLOT

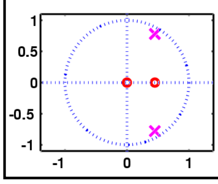


$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225 e^{-j2\hat{\omega}}}$$



$h[n]$: Decays & Oscillates

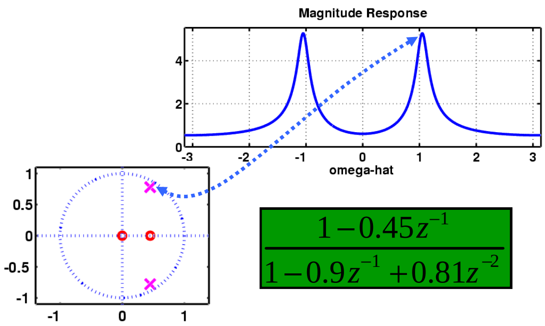




$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

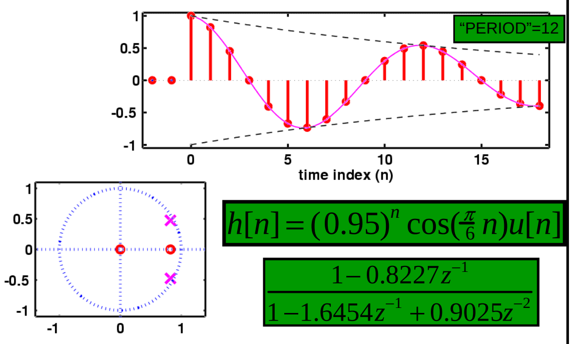
$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

Complex POLE-ZERO PLOT



$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

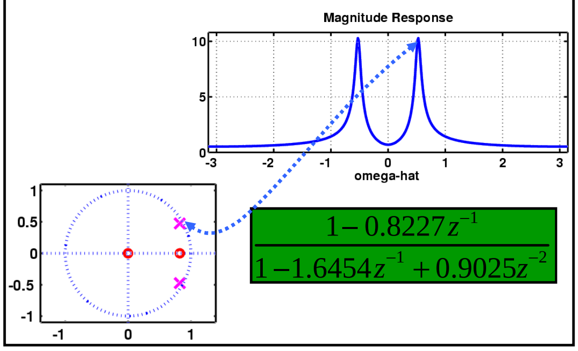
h[n]: Decays & Oscillates



$$h[n]=(0.95)^n\cos(\tfrac{\pi}{6}n)u[n]$$

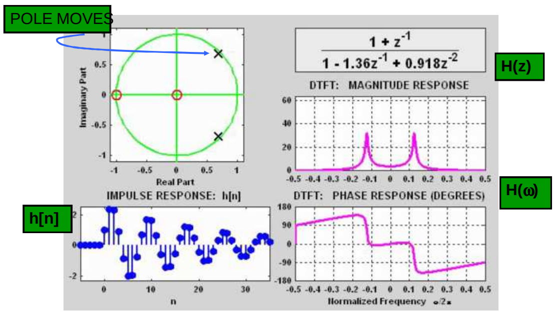
$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

Complex POLE-ZERO PLOT



$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

3 DOMAINS MOVIE: IIR



THREE INPUTS

- Given: $H(z)=\dfrac{5}{1+0.8z^{-1}}$
- Find the output, $y[n]$
  - When

$$x[n]=\cos(0.2\pi n)$$
$$x[n]=u[n]$$
$$x[n]=\cos(0.2\pi n)u[n]$$

SINUSOID ANSWER

- Given: $H(z)=\dfrac{5}{1+0.8z^{-1}}$
- The input: $x[n]=\cos(0.2\pi n)$
- Then y[n] $y[n]=M\cos(0.2\pi n+\psi)$

$$H(e^{j0.2\pi})=\frac{5}{1+0.8e^{-j0.2\pi}}=2.919e^{j0.089\pi}$$

SINUSOID Starting at *n*=0

- Given: $H(z)=\frac{5}{1+0.8z^{-1}}$
- The input: $x[n]=\cos(0.2\pi n)\,u[n]$
- Then $y[n]$ $y[n]=\Re\{h[n]*x[n]\}$ $=\Re\{5(-0.8)^n u[n]*e^{j0.2\pi n}u[n]\}$

$$y[n]=A\cos(0.2\pi n+\varphi)+B(-0.8)^n u[n]$$

SINUSOID Starting at *n*=0

$$y[n]=\Re\{5(-0.8)^n u[n]*e^{j0.2\pi n}u[n]\}$$

$$Y(z)=H(z)X(z)=\frac{5}{1+0.8z^{-1}}\,\frac{1}{1-e^{j0.2\pi}z^{-1}}$$

$$Y(z)=\frac{\frac{5}{1+1.25e^{j0.2\pi}}}{1+0.8z^{-1}}+\frac{\frac{5}{1+0.8e^{-j0.2\pi}}}{1-e^{j0.2\pi}z^{-1}}$$
$$=\frac{2.19-j0.8}{1+0.8z^{-1}}+\frac{2.81+j0.8}{1-e^{j0.2\pi}z^{-1}}$$

$H(e^{j0.2\pi})$

SINUSOID Starting at *n*=0

$$Y(z)=\frac{\frac{5}{1+1.25e^{j0.2\pi}}}{1+0.8z^{-1}}+\frac{\frac{5}{1+0.8e^{-j0.2\pi}}}{1-e^{j0.2\pi}z^{-1}}$$
$$=\frac{2.19-j0.8}{1+0.8z^{-1}}+\frac{2.81+j0.8}{1-e^{j0.2\pi}z^{-1}}$$

$H(e^{j0.2\pi})$

$$y[n]=\Re\{(2.18-j0.8)(-0.8)^n u[n]+2.92e^{j0.28}e^{j0.2\pi n}u[n]\}$$

$$y[n]=2.18(-0.8)^n u[n]+2.92\cos(0.2\pi n+0.28)u[n]$$

Transient ***Steady-State***

Step Response

$$Y(z)=H(z)X(z)=\left(\frac{5}{1+.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

Partial Fraction Expansion

$$Y(z)=\frac{A}{1+.8z^{-1}}+\frac{B}{1-z^{-1}}=\frac{(A+B)+(.8B-A)z^{-1}}{(1+.8z^{-1})(1-z^{-1})}$$

$$\Rightarrow\quad (A+B)=5 \quad\text{and}\quad (.8B-A)=0$$

$$Y(z)=\frac{A}{1+.8z^{-1}}+\frac{B}{1-z^{-1}}$$

Step Response

$$Y(z)=\frac{\frac{20}{9}}{1+.8z^{-1}}+\frac{\frac{25}{9}}{1-z^{-1}}$$

$$y[n]=\frac{20}{9}(-.8)^n u[n]+\frac{25}{9}u[n]$$

$$y[n]\to\frac{25}{9}\quad\text{as}\quad n\to\infty$$

Stability

- Nec. & suff. condition:

$$\sum_{n=-\infty}^{\infty}|h[n]|<\infty$$

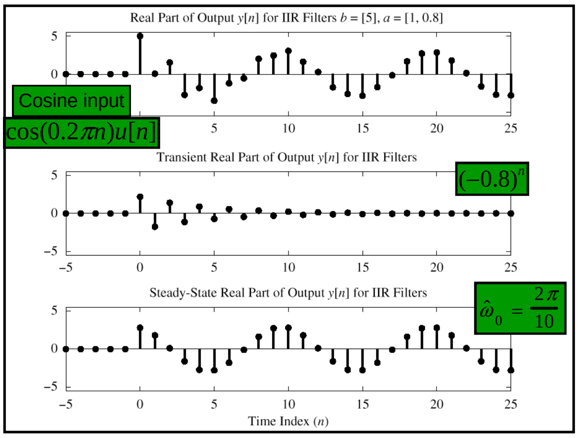
$$h[n]=b(a)^n u[n]\Leftrightarrow H(z)=\frac{b}{1-az^{-1}}$$

$$\sum_{n=0}^{\infty}|b||a|^n<\infty \text{ if } |a|<1\Rightarrow$$

Pole must be Inside unit circle

SINUSOID starting at n=0

- We'll look at an example in MATLAB
  - $\cos(0.2\pi n)$
  - Pole at –0.8, so a^n is $(-0.8)^n$
- There are two components:
  - TRANSIENT
    - Start-up region just after n=0; $(-0.8)^n$
  - STEADY-STATE
    - Eventually, $y[n]$ looks sinusoidal.
    - **Magnitude & Phase from Frequency Response**



STABILITY

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE: POLE @ z=1.1

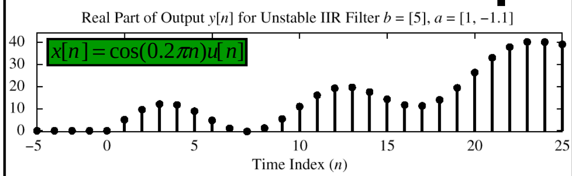
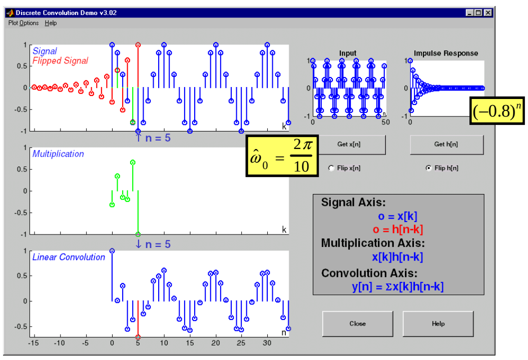


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

BONUS QUESTION

- Given: $H(z) = \dfrac{5}{1 + 0.8z^{-1}}$
- The input is $x[n] = 4\cos(\pi n - 0.5\pi)$
- Then find $y[n]$ $\quad y[n] = ?$

Transient & Steady State



CALCULATE the RESPONSE

$$x[n] = e^{j\hat{\omega}_0 n} u[n]$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$H(z)$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

Use the Z-Transform Method
And PARTIAL FRACTIONS

GENERAL INVERSE Z

PROCEDURE FOR INVERSE z-TRANSFORMATION ($M < N$)

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

$(\text{pole})^n$

SPLIT Y(z) to INVERT

- Need SUM of Terms:

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

$$= \frac{b_0}{(1 - a_1 z^{-1})(1 - e^{j\hat{\omega}_0} z^{-1})}$$

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

INVERT *Y(z)* to *y*[*n*]

- Use the Z-Transform Table

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

TWO PARTS of *y*[*n*]

- **TRANSIENT**
  - Acts Like $(\text{pole})^n$
  - Dies out ?
    - IF $|a_1|<1$

$$\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right) a_1^n u[n]$$

- **STEADY-STATE**
  - Depends on the input
  - e.g., Sinusoidal

$$\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

STEADY STATE HAPPENS

- When Transient dies out
- Limit as "n" approaches infinity
- Use Frequency Response to get Magnitude & Phase for sinusoid

$$y_{ss}[n] \rightarrow \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

NUMERICAL EXAMPLE

Example 8.12 If $b_0 = 5$, $a_1 = -0.8$, and $\hat{\omega}_0 = 2\pi/10$, the transient component is

$$y_t[n] = \left(\frac{-4}{-0.8 - e^{j0.2\pi}}\right)(-0.8)^n u[n] = 2.3351 e^{-j0.3502}(-0.8)^n u[n]$$

$$= 2.1933(-0.8)^n u[n] - j0.8012(-0.8)^n u[n]$$

Similarly, the steady-state component is

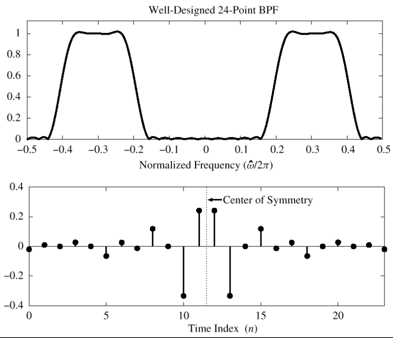
$$y_{ss}[n] = \left(\frac{5}{1 + 0.8e^{-j0.2\pi}}\right) e^{j0.2\pi n} u[n] = 2.9188 e^{j0.2781} e^{j0.2\pi n} u[n]$$

$$= 2.9188 \cos\left(\frac{2\pi}{10} n + 0.2781\right) u[n] + j\,2.9188 \sin\left(\frac{2\pi}{10} n + 0.2781\right) u[n]$$

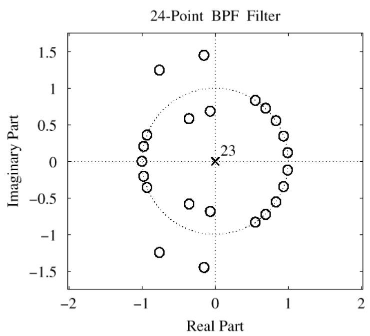
$0.089\pi = 0.2781$

REALISTIC FIR BANDPASS

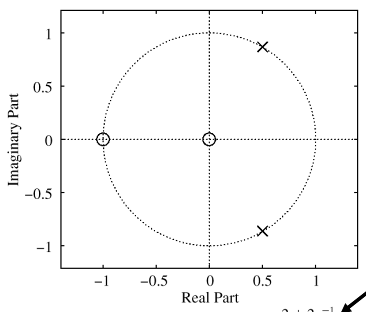
- FIR
- L = 24
- M=23
- 23 zeros



FIR BPF: 23 ZEROS



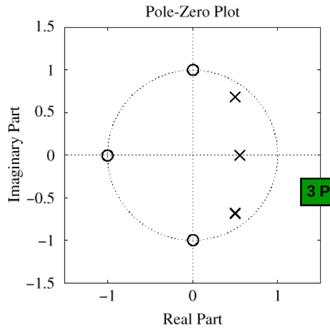
Complex POLE-ZERO PLOT



$$\frac{2z(z+1)}{z^2 - z + 1}$$

Pole-zero plot for system with $H(z) = \frac{2 + 2z^{-1}}{1 - z^{-1} + z^{-2}}$.

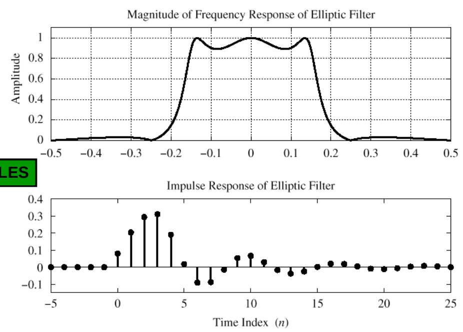
POLES & ZEROS of IIR



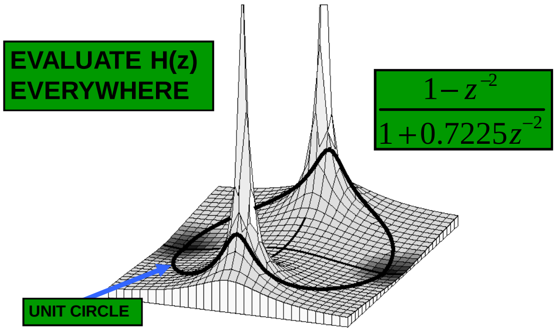
3 POLES

IIR Elliptic LPF (N=3)

3 POLES



3-D VIEW



The poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.

FLYING THRU Z-PLANE