Digital Signal Processing

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Digital Signal Processing

Lecture 17

IIR Filters: H(z) and Frequency Response

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-4 8-5 & 8-6
- Other Reading:
  - Recitation: Chapter 8, all
    - POLE-ZERO PLOTS

LECTURE OBJECTIVES

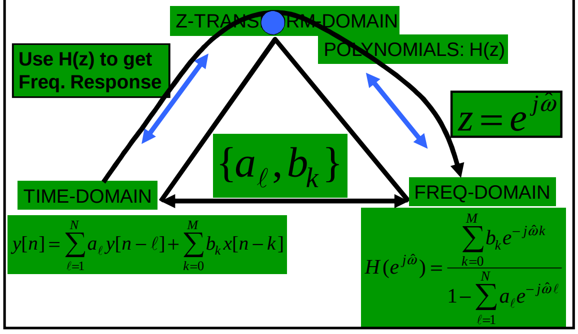
- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



$H(z)$ = z-Transform{ $h[n]$ }

- FIRST-ORDER IIR FILTER:

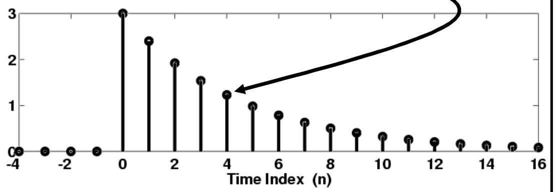
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

Typical IMPULSE Response

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



First-Order Transform Pair

$$h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

DELAY PROPERTY of $X(z)$

- DELAY in TIME<-->Multiply $X(z)$ by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

Proof: $\sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

- **READ** the FILTER COEFFS:

H(z)

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS

$$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z) = H(z)X(z)$$

- **CONVOLUTION** in TIME-DOMAIN

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) → inf

EXAMPLE: Poles & Zeros

- VALUE of *H*(*z*) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

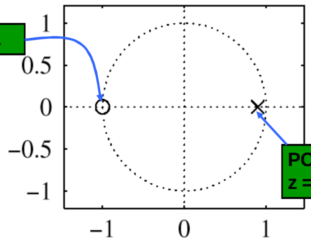
ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

POLE at z=0.8

POLE-ZERO PLOT

ZERO at z = -1



POLE at z = 0.8

$$\frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

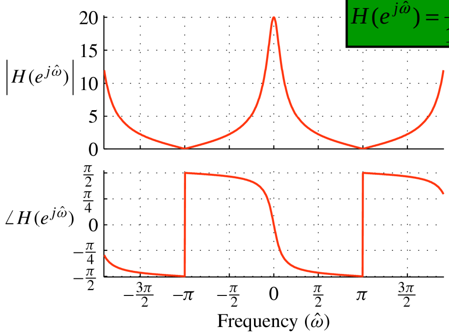
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega}=0,\quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400, \quad @\,\hat{\omega}=\pi\,?$$

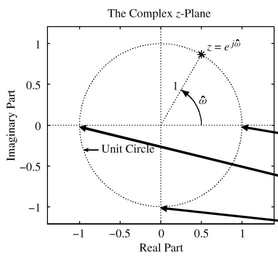
Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

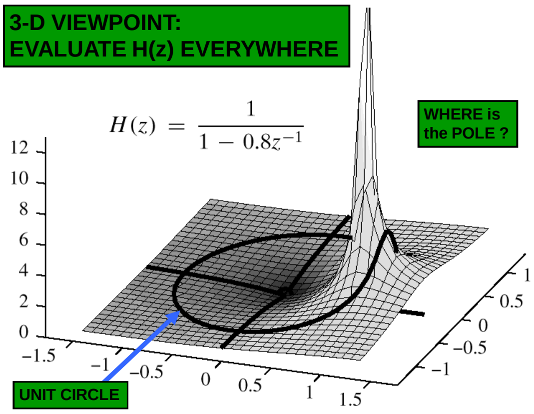
$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$

$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$

$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\frac{1}{2}\pi$

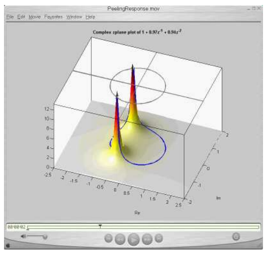
3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$



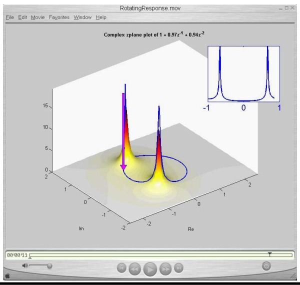
MOVIE for $H(z)$ in 3-D

- POLES to $H(z)$ to Frequency Reponse
  - TWO POLES SHOWN

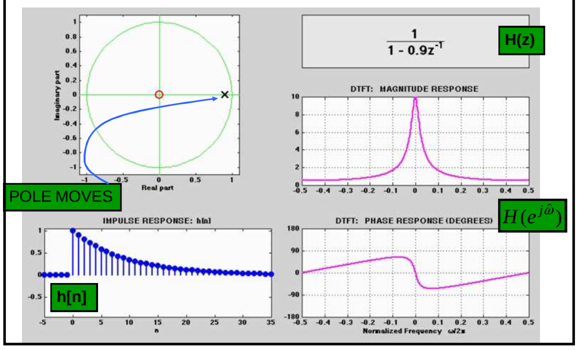


Frequency Response from $H(z)$

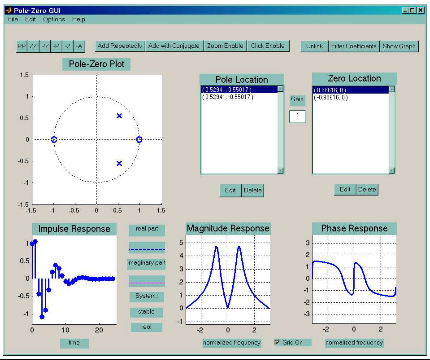
Walking around the Unit Circle



3 DOMAINS MOVIE: IIR



PeZ Demo: Pole-Zero Placing



SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

$$\text{if} \quad x[n] = e^{j\hat{\omega}n}$$

$$\text{then} \quad y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

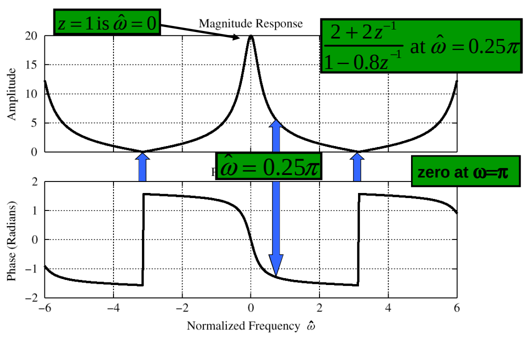
$$\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

POP QUIZ

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
  - When

$$x[n] = \cos(0.25\pi n)$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find output, $y[n]$, when
  - Evaluate at

$x[n] = \cos(0.25\pi n)$

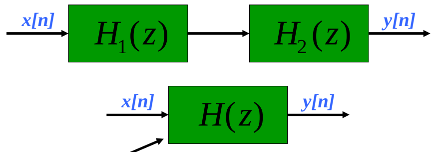
$z = e^{j0.25\pi}$

$$H(z) = \frac{2 + 2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

SUPERPOSITION

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

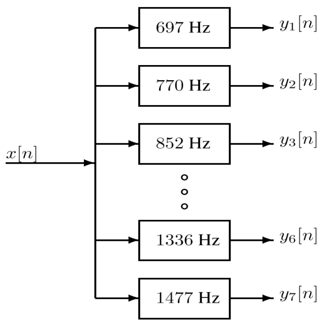
$$y[n] = y_1[n] + y_2[n]$$

where

$$y_1[n] = a_1 y_1[n-1] + b_0 x[n] \rightarrow b_0 h[n]$$

$$y_2[n] = a_1 y_2[n-1] + b_1 x[n-1] \rightarrow b_1 h[n-1]$$

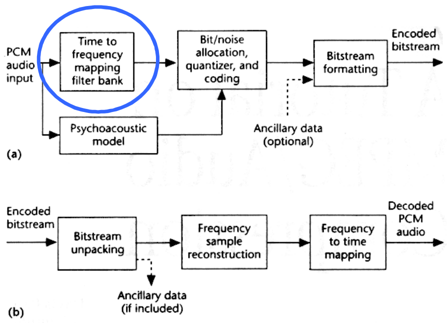
FILTER BANK in LAB (BPFs)



CONSTANT BANDWIDTH
- if L is same
- 1/L controls BW

VARIABLE BANDWIDTH ?
- MP3
- match hearing
- Change L

MP-3 AUDIO CODING



H(z) from Linearity

- FIRST-ORDER IIR with 2 FIR terms:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0\, a_1^n u[n] + b_1\, a_1^{n-1} u[n-1]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

INTERPRET ROOTS

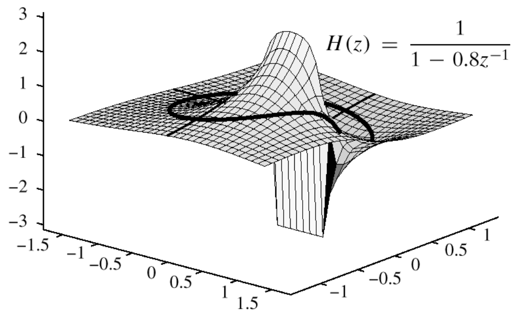
- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 z + b_1}{z - a_1}$$

$$H(z)\big|_{z=-(b_1/b_0)} = 0$$ **ZERO**

$$H(z)\big|_{z=a_1} \to \infty$$ **POLE**

PHASE from 3-D PLOT



FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \to H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\hat{\omega}=0, \quad |H(e^{j\hat{\omega}})|^2 = \frac{16}{0.04} = 400 \qquad @\hat{\omega}=\pi?$$

THREE DOMAINS

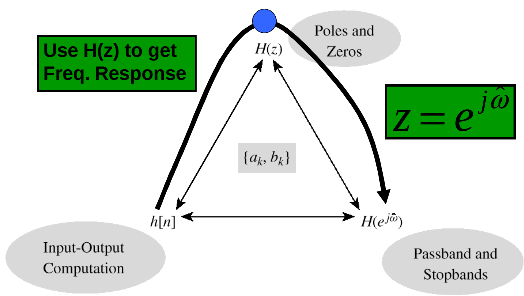


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k, b_k\}$ play a central role.

FREQ. RESPONSE FORMULA

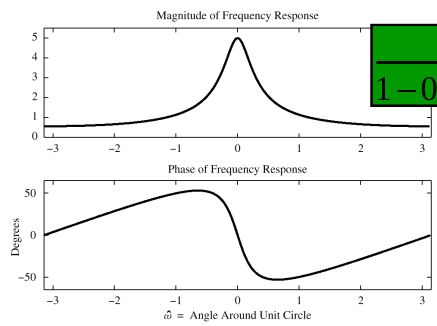
$$H(z) = \frac{1}{1-0.8z^{-1}} \to H(e^{j\hat{\omega}}) = \frac{1}{1-0.8e^{-j\hat{\omega}}}$$

$$|H(e^{j\hat{\omega}})|^2 = \left|\frac{1}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{1}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{1}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{1}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{1}{1.64-1.6\cos\hat{\omega}}$$

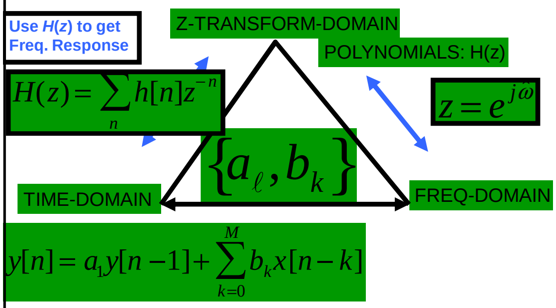
$$@\hat{\omega}=0, \quad |H(e^{j\hat{\omega}})|^2 = \frac{1}{0.04} = 25 \qquad @\hat{\omega}=\pi?$$

FREQ. RESPONSE from *H*(*z*)



$$\frac{1}{1-0.8e^{-j\hat{\omega}}}$$

THREE DOMAINS



POP QUIZ

- Given: $H(z)=\dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
  - When

$$x[n]=\cos(0.25\pi n)$$

POP QUIZ: Invert Z

- Given:

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{2}{1-0.8z^{-1}}+\frac{2z^{-1}}{1-0.8z^{-1}}$$

- Find the Impulse Response, $h[n]$
  - Use the DELAY PROPERTY

$$h[n]=2(0.8)^n u[n]+2(0.8)^{n-1}u[n-1]$$