Digital Signal Processing

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Lecture 16

IIR Filters: Feedback and H(z)

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READING ASSIGNMENTS

- This Lecture:
    - Chapter 8, Sects. 8-1, 8-2 & 8-3
- Other Reading:
    - Recitation: Ch. 8, Sects 8-1 thru 8-4
        - POLES & ZEROS
    - Next Lecture: Chapter 8, Sects. 8-4 8-5 & 8-6

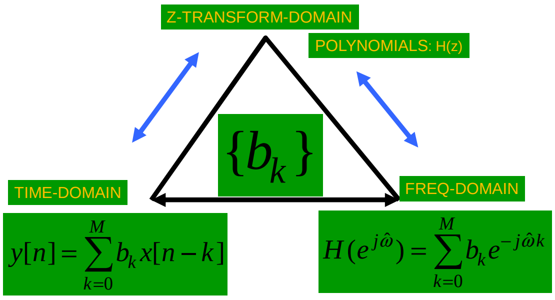
LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
    - Define **IIR** DIGITAL Filters
    - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n]=\sum_{\ell=1}^{N}a_\ell y[n-\ell]+\sum_{k=0}^{M}b_k x[n-k]$$

- Show how to compute the output $y[n]$
    - FIRST-ORDER CASE (N=1)
    - Z-transform: Impulse Response $h[n]$ ←→ $H(z)$

THREE DOMAINS



Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$\{b_k\}$

TIME-DOMAIN

$$y[n]=\sum_{k=0}^{M}b_k x[n-k]$$

FREQ-DOMAIN

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M}b_k e^{-j\hat{\omega}k}$$

Quick Review: Delay by n_d

$$y[n] = x[n-n_d]$$

IMPULSE RESPONSE $h[n] = \delta[n-n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$

Quick Review: L-pt Averager

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE $h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$

SYSTEM FUNCTION $H(z) = \sum_{n=0}^{L-1} \tfrac{1}{L} z^{-n}$

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, $h[n]$
  - INFINITELY LONG
  - **IIR** Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

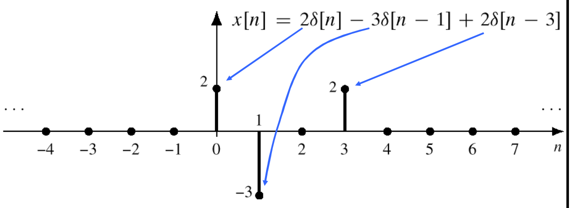
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$



COMPUTE *y*[*n*]

- FEEDBACK DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

- NEED y[-1] to get started

$$\boxed{y[0] = 0.8y[-1] + 5x[0]}$$

AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE *y*[*n*]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$\underset{\blacktriangle}{y[2]} = 0.8\underset{\blacktriangle}{y[1]} + 5\underset{\blacktriangle}{x[2]} = 0.8(-7) + 5(0) = -5.6$$

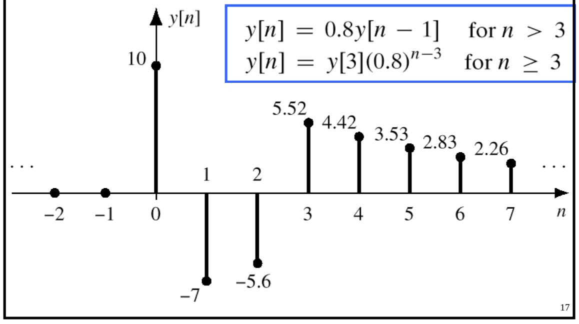
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT *y*[*n*]



$$\boxed{\begin{aligned} y[n] &= 0.8y[n-1] \quad \text{for } n > 3 \\ y[n] &= y[3](0.8)^{n-3} \quad \text{for } n \geq 3 \end{aligned}}$$

IMPULSE RESPONSE

$$\boxed{h[n] = a_1 h[n-1] + b_0 \delta[n]}$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\boxed{u[n] = 1, \quad \text{for } n \geq 0}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

IMPULSE RESPONSE

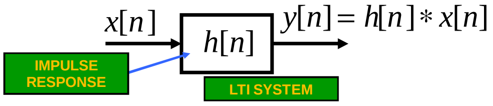
- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n]}$$

- Find h[n]

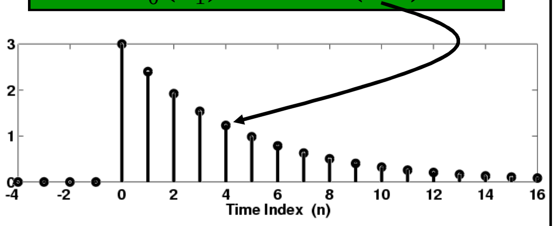
$$\boxed{h[n] = 3(0.8)^n u[n]}$$

- **CONVOLUTION** in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: *h*[*n*]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of *H*(*z*)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$
$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

H(*z*) = z-Transform{ *h*[*n*] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

H(*z*) = z-Transform{ *h*[*n*] }

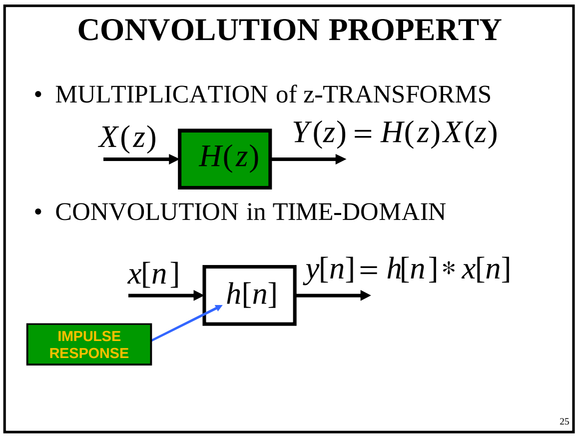
- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$\boxed{h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]}$$

z^{-1} is a shift

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$



CONVOLUTION PROPERTY

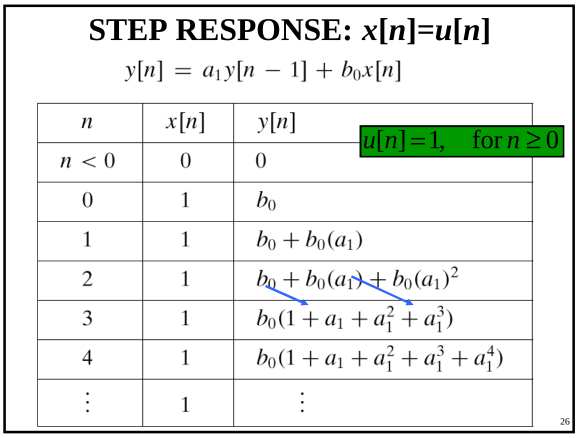
- MULTIPLICATION of z-TRANSFORMS

$$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z) = H(z)X(z)$$

- CONVOLUTION in TIME-DOMAIN

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

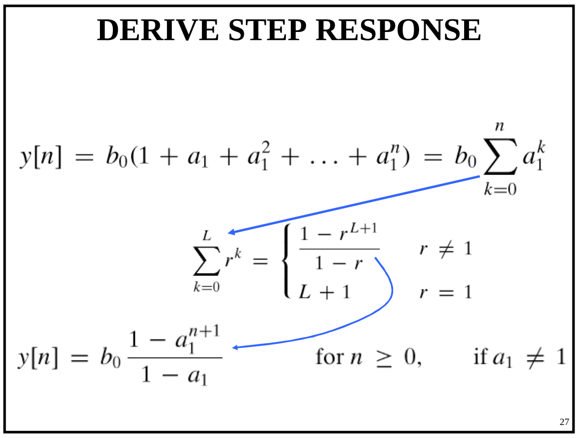


STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n] = 1, \quad \text{for } n \geq 0$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

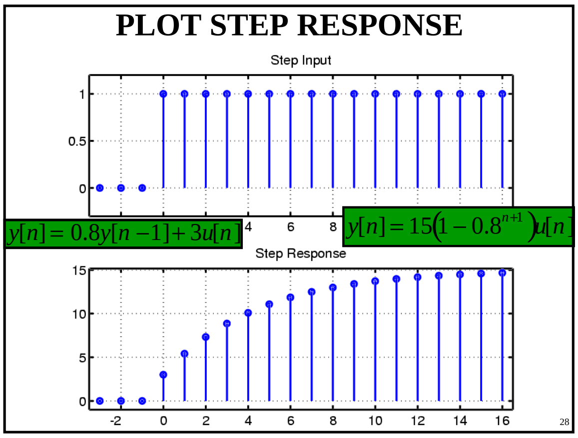


DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

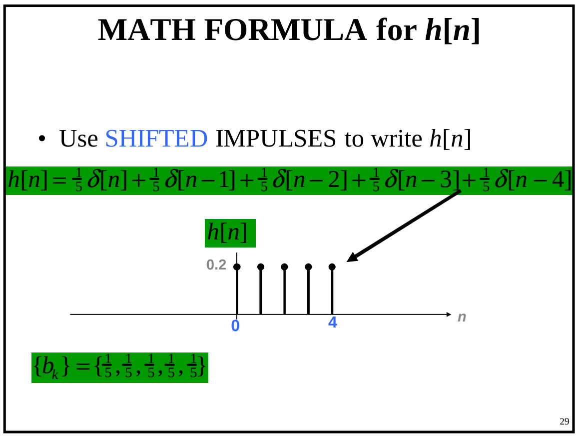
$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \quad \text{if } a_1 \neq 1$$



PLOT STEP RESPONSE

$y[n] = 0.8y[n-1] + 3u[n]$

$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$

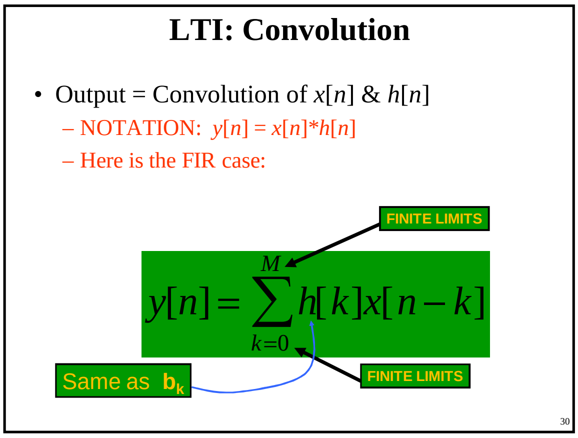


MATH FORMULA for *h*[*n*]

- Use SHIFTED IMPULSES to write $h[n]$

$$h[n] = \tfrac{1}{5}\delta[n] + \tfrac{1}{5}\delta[n-1] + \tfrac{1}{5}\delta[n-2] + \tfrac{1}{5}\delta[n-3] + \tfrac{1}{5}\delta[n-4]$$

$\{b_k\} = \{\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\}$



LTI: Convolution

- Output = Convolution of $x[n]$ & $h[n]$
  - NOTATION: $y[n] = x[n]*h[n]$
  - Here is the FIR case:

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Same as $\mathbf{b_k}$

FINITE LIMITS

L-pt RUNNING AVG *H*(*z*)

$$H(z)=\sum_{k=0}^{L-1}\tfrac{1}{L}z^{-k}=\frac{1-z^{-L}}{L(1-z^{-1})}=\frac{z^{L}-1}{Lz^{L-1}(z-1)}$$

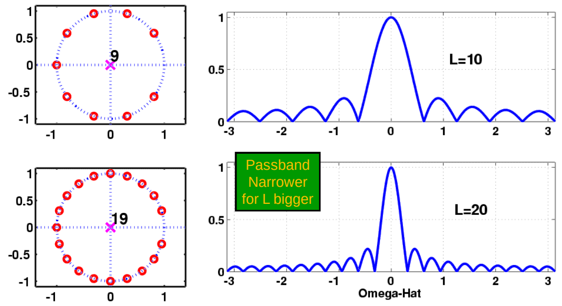
$$z^{L}-1=0 \quad\Rightarrow\quad z^{L}=1=e^{j2\pi k}$$

$$z=e^{j(2\pi/L)k} \quad \text{for } k=1,2,\ldots L\text{-}1$$

ZEROS on UNIT CIRCLE

(*z*-1) in denominator cancels *k*=0 term

FILTER DESIGN: CHANGE *L*



Passband Narrower for L bigger

H(*z*) = z-Transform{ *h*[*n*] }

- FIRST-ORDER CASE:

$$b_0=1,\ a_1=a$$

$$h[n]=a^{n}u[n]$$

$$H(z)=\sum_{n=0}^{\infty}a^{n}z^{-n}=\frac{1}{1-az^{-1}} \qquad \boxed{|a|<z.}$$

$$\boxed{a^{n}u[n] \quad\Longleftrightarrow\quad \frac{1}{1-az^{-1}}}$$

POP QUIZ

- Given: $H(\hat{\omega})=e^{-j\hat{\omega}}\cos(\hat{\omega})$
- Plot the Magnitude and Phase
- Find the output, *y*[*n*]
  - When

$$x[n]=\cos(0.25\pi n)$$

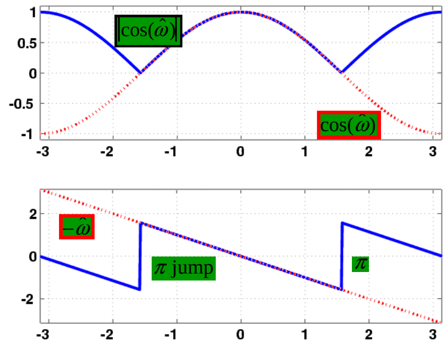
POP QUIZ: MAG & PHASE

- Given: $H(\hat{\omega})=e^{-j\hat{\omega}}\cos(\hat{\omega})$
- Derive Magnitude and Phase

$$|H(\hat{\omega})|=|e^{-j\hat{\omega}}||\cos(\hat{\omega})|=|\cos(\hat{\omega})|$$

$$\angle H(\hat{\omega})=\begin{cases}-\hat{\omega} & \cos(\hat{\omega})\geq 0\\ -\hat{\omega}+\pi & \text{if}\cos(\hat{\omega})<0\end{cases}$$

Ans: FREQ RESPONSE



POP QUIZ : Answer #2

- Find $y[n]$ when

$$\boxed{x[n] = \cos(0.25\pi n)}$$

$$y[n] = |H|\cos(0.25\pi n + \angle H)$$

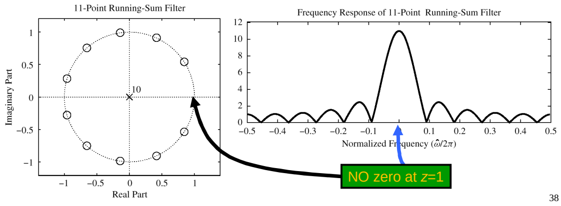
$$\boxed{= 0.707\cos(0.25\pi n - \tfrac{\pi}{4})}$$

$$\boxed{H(\hat{\omega}) = e^{-j\hat{\omega}}\cos(\hat{\omega}) \quad \boxed{\text{at } \hat{\omega} = \tfrac{\pi}{4}} \\ H(\tfrac{\pi}{4}) = e^{-j\pi/4}\cos(\tfrac{\pi}{4}) = 0.707e^{-j\pi/4}}$$

11-pt RUNNING SUM *H*(*z*)

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1})\cdots(1 - e^{j20\pi/11}z^{-1})$$



NO zero at *z*=1

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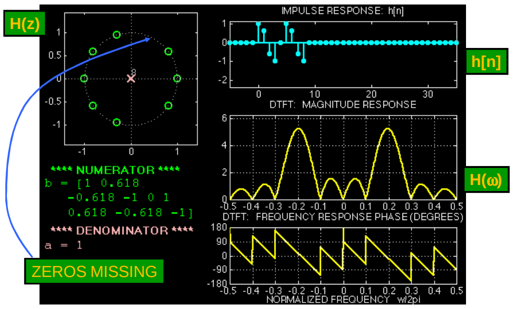
3 DOMAINS MOVIE: FIR



```
**** NUMERATOR ****
b = [1 0.618
   -0.618 -1 0 1
   0.618 -0.618 -1]
**** DENOMINATOR ****
a = 1
```

ZEROS MISSING

L-pt RUNNING AVG: Step Response

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L}\, x[n-k]$$

STEP RESPONSE

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L}\, u[n-k] = \begin{cases} \frac{n+1}{L} & n = 0,1,2,\ldots L-1 \\ 1 & n \geq L \\ 0 & n < 0 \end{cases}$$