Digital Signal Processing

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Digital Signal Processing

Lecture 15

Zeros of H(z) and the Frequency Domain

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 7, Section 7-6 to end
- Other Reading:
  - Recitation & Lab: Chapter 7
    - ZEROS (and POLES)
  - Next Lecture:Chapter 8

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LECTURE OBJECTIVES

- ZEROS and POLES
- Relate H(z) to FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE DOMAINS:
  - Show Relationship for FIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter (Find b_k)
  - Reject completely 0.7**π**, 0.8**π**, and 0.9**π**
    - This is NULLING
  - Estimate the filter length needed to accomplish this task. How many b_k ?
- Z POLYNOMIALS provide the TOOLS

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Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $H(z)=\sum_{n} h[n] z^{-n}$

APPLIES to Any SIGNAL

- EXAMPLE:

$\{h[n]\}=\{2,0,-3,0,2\}$

$$H(z)=2z^{-0}+0z^{-1}-3z^{-2}+0z^{-3}+2z^{-4}$$
$$=2-3z^{-2}+2z^{-4}$$
$$=2-3(z^{-1})^2+2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

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CONVOLUTION PROPERTY

- Convolution in the n-domain
  - SAME AS
- Multiplication in the z-domain

$$y[n]=h[n]*x[n] \quad \leftrightarrow \quad Y(z)=H(z)X(z)$$

MULTIPLY z-TRANSFORMS

$$y[n]=x[n]*h[n]$$
$$=\sum_{k=0}^{M} h[k]x[n-k]$$

FIR Filter

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CONVOLUTION EXAMPLE

$x[n]$ → $H(z)$ → $y[n]$

$x[n]=\delta[n-1]+2\delta[n-2]$

$h[n]=\delta[n]-\delta[n-1]$

$y[n]=x[n]*h[n]$

$X(z)=z^{-1}+2z^{-2}$

$H(z)=1-z^{-1}$

$$Y(z)=(z^{-1}+2z^{-2})(1-z^{-1})=z^{-1}+z^{-2}-2z^{-3}$$

$$y[n]=\delta[n-1]+\delta[n-2]-2\delta[n-3]$$

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THREE DOMAINS

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$\{b_k\}$

TIME-DOMAIN

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]$$

FREQ-DOMAIN

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

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FREQUENCY RESPONSE ?

- Same Form:

$$z=e^{j\hat{\omega}}$$

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}})=\sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

z – Domain

$$H(z)=\sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

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ANOTHER ANALYSIS TOOL

- z-Transform POLYNOMIALS are EASY !
  - ROOTS, FACTORS, etc.
- **ZEROS and POLES: where is $H(z) = 0$?**
- The z-domain is COMPLEX
  - $H(z)$ is a COMPLEX-VALUED function of a COMPLEX VARIABLE z.

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ZEROS of $H(z)$

- Find z, where $H(z)$=0

$$H(z) = 1 - \tfrac{1}{2}z^{-1}$$

$$1 - \tfrac{1}{2}z^{-1} = 0\ ?$$

$$z - \tfrac{1}{2} = 0$$

$$\text{Zero at} : z = \tfrac{1}{2}$$

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ZEROS of $H(z)$

- Find z, where $H(z)$=0
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots} : z = 1, \tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2} \qquad e^{\pm j\pi/3}$$

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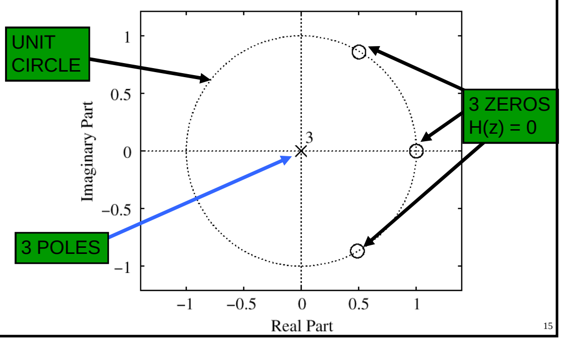
PLOT ZEROS in z-DOMAIN



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POLES of H(z)

- Find z, where $H(z) \to \infty$
  - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

$$\text{Three Poles at} : z = 0$$

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FREQ. RESPONSE from ZEROS

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- Relate $H(z)$ to FREQUENCY RESPONSE
- EVALUATE $H(z)$ on the **UNIT CIRCLE**
  - ANGLE is same as FREQUENCY

$$z = e^{j\hat{\omega}} \qquad \text{(as } \hat{\omega} \text{ varies)}$$
$$\text{defines a CIRCLE, radius} = 1$$

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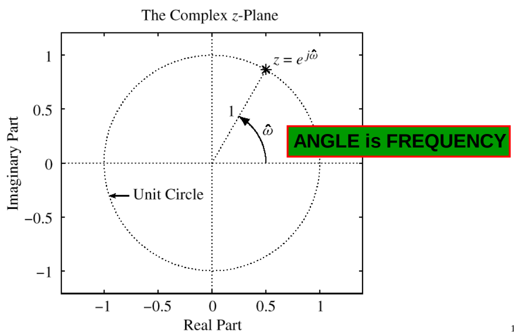
$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



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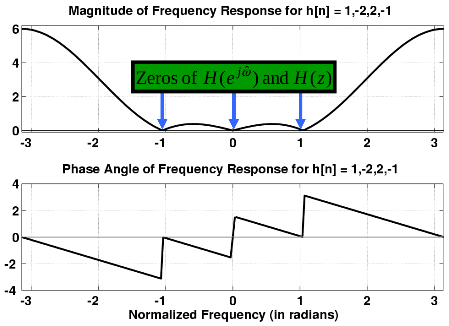
FIR Frequency Response



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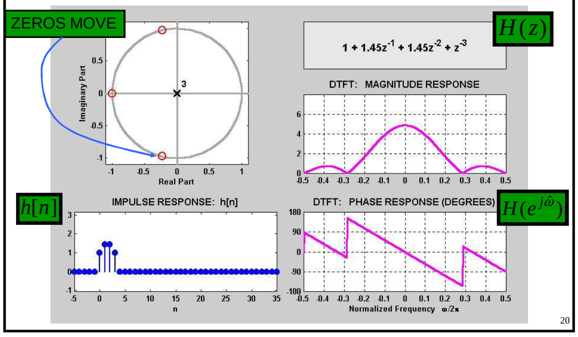
3 DOMAINS MOVIE: FIR



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NULLING PROPERTY of $H(z)$

- When $H(z)$=0 on the unit circle.
  - Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

$x[n]$ → $H(z)$ → $y[n]$

$$H(e^{j\pi/3}) = ?$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

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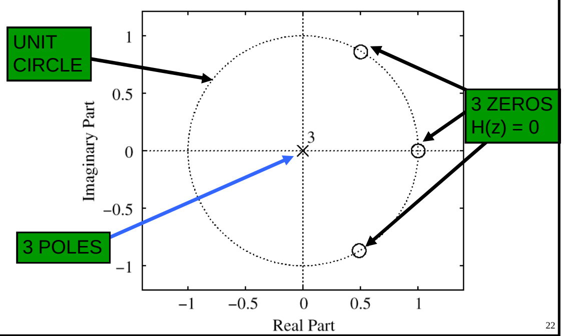
PLOT ZEROS in z-DOMAIN



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NULLING PROPERTY of $H(z)$

- Evaluate $H(z)$ at the input "frequency"

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) - (-1))$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

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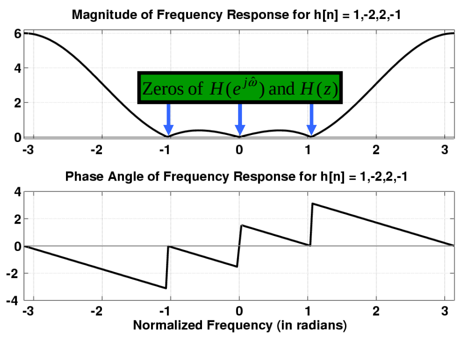
FIR Frequency Response



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DESIGN PROBLEM

- Example:
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  - Estimate the filter length needed to accomplish this task. How many b_k ?
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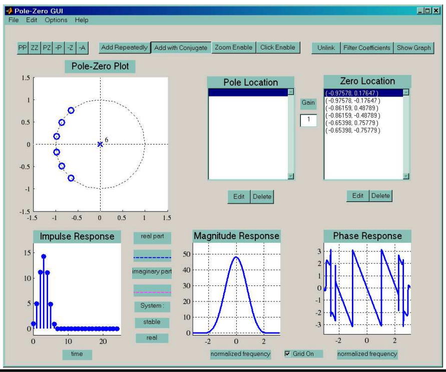
PeZ Demo: Zero Placing



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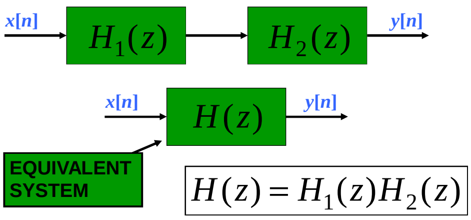
CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

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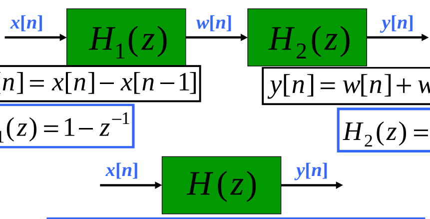
CASCADE EXAMPLE



$$w[n] = x[n] - x[n-1]$$

$$y[n] = w[n] + w[n-1]$$

$$H_1(z) = 1 - z^{-1}$$

$$H_2(z) = 1 + z^{-1}$$

$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n-2]$$

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L-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{L-1} \frac{1}{L} z^{-k} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^L - 1}{L z^{L-1}(z-1)}$$

$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \qquad \text{for } k = 1, 2, \ldots L-1$$

ZEROS on UNIT CIRCLE

(z-1) in denominator cancels k=0 term

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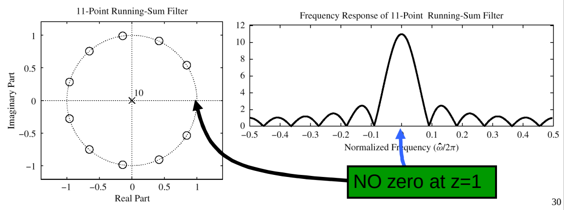
11-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1}) \cdots (1 - e^{j20\pi/11}z^{-1})$$



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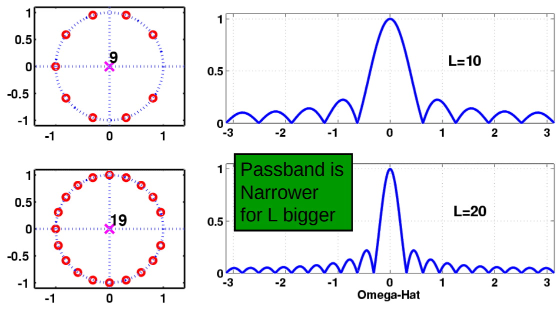
FILTER DESIGN: CHANGE L



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NULLING FILTER

- PLACE ZEROS to make $y[n] = 0$

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$ ← 3 ZEROS $H(z) = 0$

the output resulting from each of the following three signals will be zero:

$H(z_1) = 0$	$x_1[n] = (z_1)^n = 1$	$y_1[n] = 0$
$H(z_2) = 0$	$x_2[n] = (z_2)^n = e^{j\pi n/3}$	$y_2[n] = 0$
$H(z_3) = 0$	$x_3[n] = (z_3)^n = e^{-j\pi n/3}$	$y_3[n] = 0$

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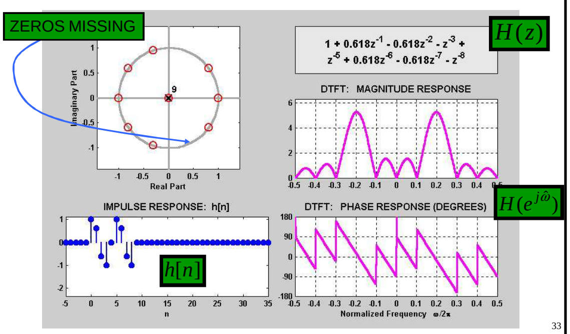
3 DOMAINS MOVIE: FIR



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POP QUIZ: MAG & PHASE

- Given: $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}} \cos(\hat{\omega})$
- Derive Magnitude and Phase

$$|H(e^{j\hat{\omega}})| = |e^{-j\hat{\omega}}| \cdot |\cos(\hat{\omega})| = |\cos(\hat{\omega})|$$

$$\angle H(e^{j\hat{\omega}}) = \begin{cases} -\hat{\omega} & \cos(\hat{\omega}) \ge 0 \\ -\hat{\omega} + \pi & \cos(\hat{\omega}) < 0 \end{cases}$$

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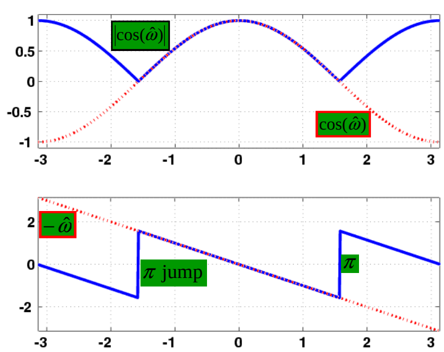
Ans: FREQ RESPONSE



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POP QUIZ : Answer #2

- Find $y[n]$ when

$$x[n] = \cos(0.25\pi n)$$

$$y[n] = |H| \cos(0.25\pi n + \angle H)$$

$$= 0.707 \cos(0.25\pi n - \tfrac{\pi}{4})$$

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}} \cos(\hat{\omega}) \quad \text{at } \hat{\omega} = \tfrac{\pi}{4}$$

$$H(e^{j\pi/4}) = e^{-j\pi/4} \cos(\tfrac{\pi}{4}) = 0.707 e^{-j\pi/4}$$

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CHANGE in NOTATION

- Relate $H(z)$ to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

NEW NOTATION for FREQUENCY RESPONSE

$$H(\hat{\omega}) \leftrightarrow H(e^{j\hat{\omega}})$$

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