Digital Signal Processing

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Digital Signal Processing

Lecture 8

Sampling & Aliasing

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License Info for SPFirst Slides

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READING ASSIGNMENTS

- This Lecture:
  - Chap 4, Sections 4-1 and 4-2
    - Replaces Ch 4 in DSP First, pp. 83-94
- Other Reading:
  - Recitation: Strobe Demo (Sect 4-3)
  - Next Lecture: Chap. 4 Sects. 4-4 and 4-5

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LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - **Sampling Theorem**
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, $x[n]$
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

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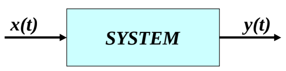
SYSTEMS Process Signals



- PROCESSING GOALS:
  - Change $x(t)$ into $y(t)$
    - For example, more BASS
  - Improve $x(t)$, e.g., image deblurring
  - Extract Information from $x(t)$

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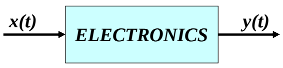
System IMPLEMENTATION

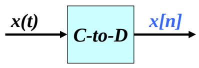
- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps



- DIGITAL/MICROPROCESSOR
  - Convert $x(t)$ to numbers stored in memory

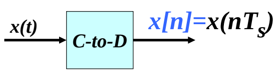
SAMPLING $x(t)$

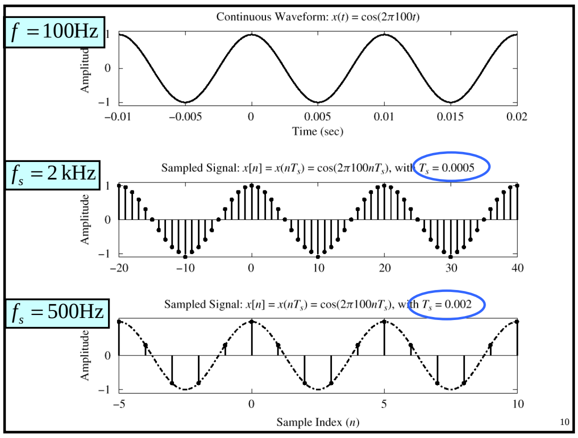
- SAMPLING PROCESS
  - Convert $x(t)$ to numbers $x[n]$
  - "n" is an integer; $x[n]$ is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - $T_s = 125$ microsec → $f_s = 8000$ samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$





SAMPLING THEOREM

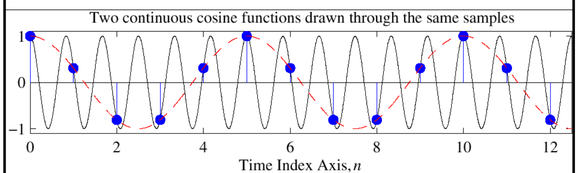
- HOW OFTEN ?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer

$\cos(0.4\pi n) = \cos(2.4\pi n)$

STORING DIGITAL SOUND

- ***x[n]*** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 × (16/8) × 60 × 44100 = 10.584 Mbytes

DISCRETE-TIME SINUSOID

- Change x(t) into x[n]   DERIVATION

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

DIGITAL FREQUENCY $\hat{\omega}$

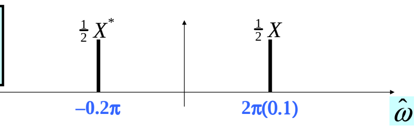
- $\hat{\omega}$ VARIES from **0** to **2π**, as *f* varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
  - DIGITAL FREQUENCY is NORMALIZED

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

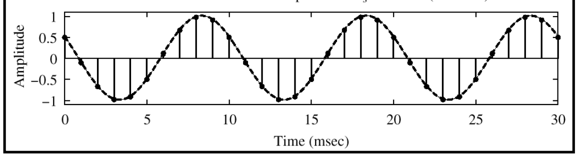
SPECTRUM (DIGITAL)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\,\text{kHz}$



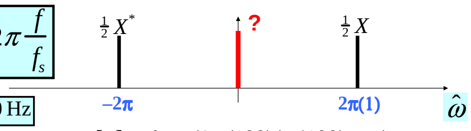
$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



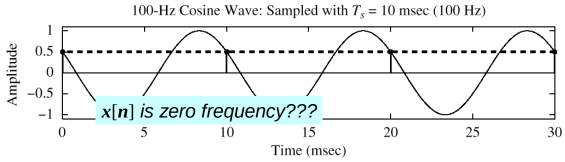
SPECTRUM (DIGITAL) ???

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 100\text{ Hz}$



$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$



The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**
- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega} n + \varphi) = A\cos((\hat{\omega} + 2\pi)n + \varphi)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$x_1(t) = \cos(400\pi t)$ sampled at $f_s = 1000\,\text{Hz}$

$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$

$x_2(t) = \cos(2400\pi t)$ sampled at $f_s = 1000\,\text{Hz}$

$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$

$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$

$\Rightarrow x_2[n] = x_1[n]$ $\quad 2400\pi - 400\pi = 2\pi(1000)$

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ALIASING DERIVATION–2

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$ $\quad \boxed{t \leftarrow \frac{n}{f_s}}$

and we want : $x[n] = A\cos(\hat{\omega} n + \varphi)$

then : $\hat{\omega} = \dfrac{2\pi(f + \ell f_s)}{f_s} = \dfrac{2\pi f}{f_s} + \dfrac{2\pi \ell f_s}{f_s}$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}\boxed{+2\pi\ell}$$

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ALIASING CONCLUSIONS

- ADDING f_s or $2f_s$ or $-f_s$ to the FREQ of $x(t)$ gives exactly the same $x[n]$
  - The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES
- GIVEN $x[n]$, WE CAN'T DISTINGUISH f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$

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NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}\boxed{+2\pi\ell}$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

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SPECTRUM for $x[n]$

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS

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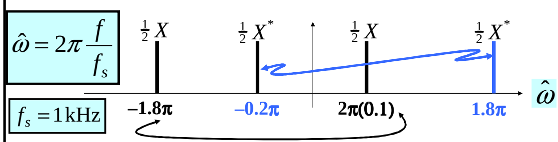
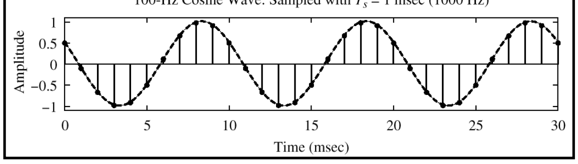
SPECTRUM (MORE LINES)

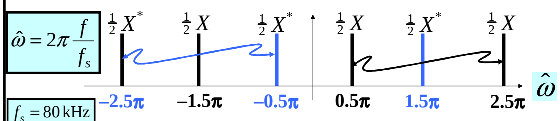


$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$

100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)

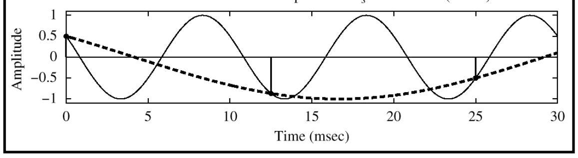


SPECTRUM (ALIASING CASE)

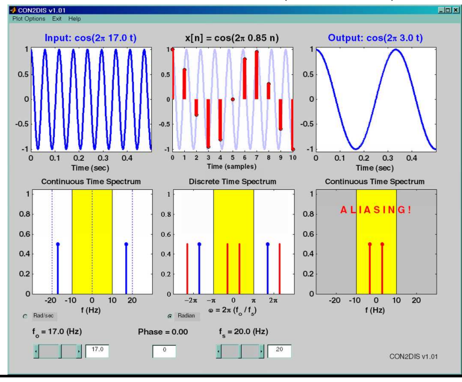


$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

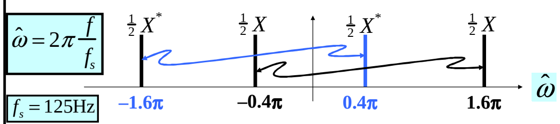
100-Hz Cosine Wave: Sampled with T_s = 12.5 msec (80 Hz)



SAMPLING GUI (con2dis)

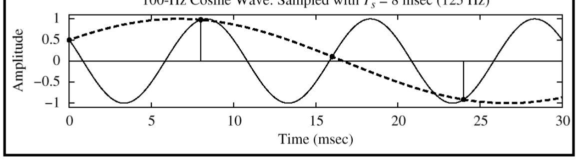


SPECTRUM (FOLDING CASE)

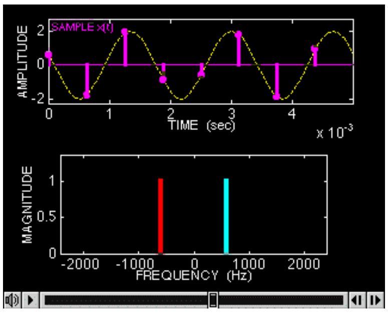


$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 8 msec (125 Hz)



SAMPLING DEMO (Chap. 4)



ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$

and we substitute: $t \leftarrow \frac{n}{f_s}$

then: $x[n] = A\cos(2\pi(f + \ell f_s)\frac{n}{f_s} + \varphi)$

or, $x[n] = A\cos(2\pi\frac{f}{f_s}n + 2\pi\ell n + \varphi)$

ALIASING DERIVATION–2

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$ $\quad \boxed{t \leftarrow \frac{n}{f_s}}$

and we want: $x[n] = A\cos(\hat{\omega}n + \varphi)$

then: $\hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}\boxed{+2\pi\ell}$$

FOLDING DERIVATION

- Negative Freqs can give the same $\hat{\omega}$

$$x(t) = A\cos(2\pi(\underline{-f + \ell f_s})t - \varphi)$$

$$x[n] = x(nT_s) = A\cos(2\pi(-f + \ell f_s)nT_s - \varphi)$$

$$x[n] = A\cos((-2\pi f T_s)n + (2\pi \ell f_s T_s)n - \varphi)$$

$$x[n] = A\cos((2\pi f T_s)n - \underline{2\pi\ell n} + \varphi) \qquad \cos(-\theta) = \cos\theta$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

SAME DIGITAL SIGNAL

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FOLDING (a type of ALIASING)

- MANY $x(t)$ give IDENTICAL $x[n]$
- CAN'T TELL f_o FROM $(f_s - f_o)$
  - Or, $(2f_s - f_o)$ or, $(3f_s - f_o)$
- EXAMPLE:
  - $y(t)$ has 1000 Hz component
  - SAMPLING FREQ = 1500 Hz
  - WHAT is the "FOLDED" ALIAS ?

$$-1000 + 1500 \rightarrow 500$$

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DIGITAL FREQ $\hat{\omega}$ AGAIN

Normalized Radian Frequency

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} + 2\pi\ell$$

ALIASING

$$\boxed{\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi\ell}$$

FOLDED ALIAS

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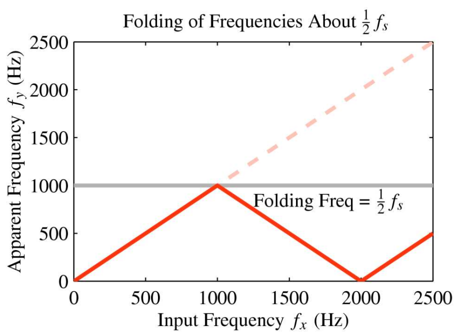
FOLDING DIAGRAM



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FOLDING DIAGRAM

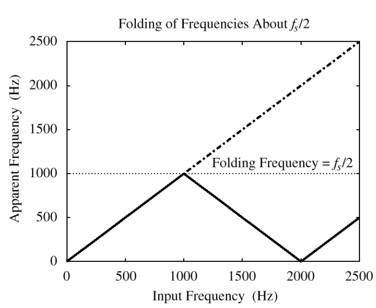

Figure 4.4 Folding of a sinusoid sampled at $f_s = 2000$ samples/sec. The apparent frequency is the lowest frequency of a sinusoid that has exactly the same samples as the input sinusoid.

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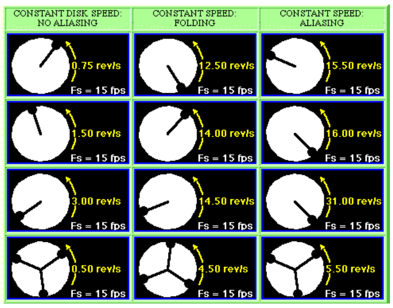
STROBE DEMO (Synthetic)



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ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$$x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(2\pi(f + \ell f_s)nT_s + \varphi)$$

$$x[n] = A\cos((2\pi f T_s)n + (2\pi \ell f_s T_s)n + \varphi)$$

$$x[n] = A\cos((2\pi f T_s)n + \underline{2\pi \ell n} + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi) \qquad \hat{\omega} = 2\pi f T_s = \frac{2\pi f}{f_s}$$

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