

Digital Signal Processing

Prof. Nizamettin AYDIN

naydin@yildiz.edu.tr

<http://www.yildiz.edu.tr/~naydin>

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Digital Signal Processing

Lecture 7

Fourier Series & Spectrum

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READING ASSIGNMENTS

- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: a_k for Fourier Series
- Other Reading:
 - **Next Lecture: Sampling**

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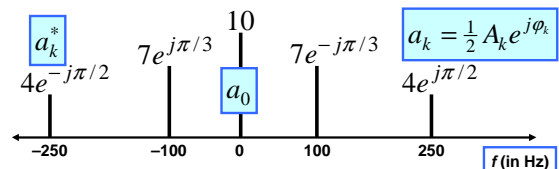
LECTURE OBJECTIVES

- **ANALYSIS** via Fourier Series
 - For **PERIODIC** signals: $x(t+T_0) = x(t)$
- $$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$
- **SPECTRUM** from Fourier Series
 - a_k is Complex Amplitude for k -th Harmonic

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SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

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Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

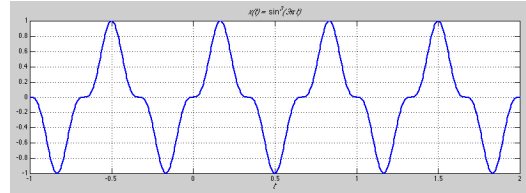
PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

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Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

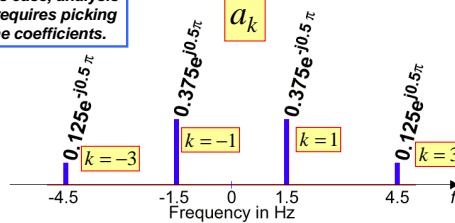
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Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



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STRATEGY: $x(t) \rightarrow a_k$

ANALYSIS

- Get representation from the signal
- Works for **PERIODIC** Signals

Fourier Series

- Answer is: an **INTEGRAL** over one period

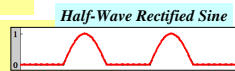
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

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FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq \pm 1)$$

$$\begin{aligned} a_k &= \frac{1}{T_0} \int_0^{T_0/2} \sin\left(\frac{2\pi}{T_0}t\right) e^{-j(2\pi/T_0)kt} dt \\ &= \frac{1}{T_0} \int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt \\ &= \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t} dt - \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t} dt \\ &= \frac{e^{-j(2\pi/T_0)(k-1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k-1))} \Big|_0^{T_0/2} - \frac{e^{-j(2\pi/T_0)(k+1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k+1))} \Big|_0^{T_0/2} \end{aligned}$$



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FS: Rectified Sine Wave $\{a_k\}$

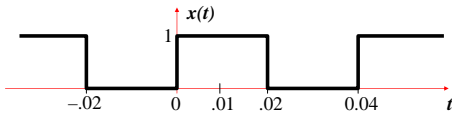
$$\begin{aligned} a_k &= \frac{e^{-j(2\pi/T_0)(k-1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k-1))} \Big|_0^{T_0/2} - \frac{e^{-j(2\pi/T_0)(k+1)T_0/2}}{j2T_0(-j(2\pi/T_0)(k+1))} \Big|_0^{T_0/2} \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1 \right) \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j\pi(k-1)} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j\pi(k+1)} - 1 \right) \\ &= \begin{cases} 0 & k \text{ odd} \\ \pm \frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases} \end{aligned}$$

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SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



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FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

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DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

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Fourier Coefficients a_k

- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

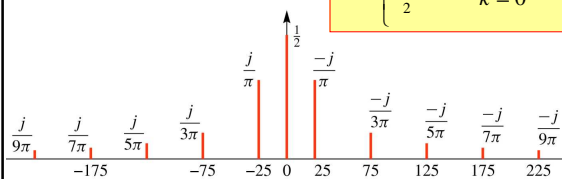
$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

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Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



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Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

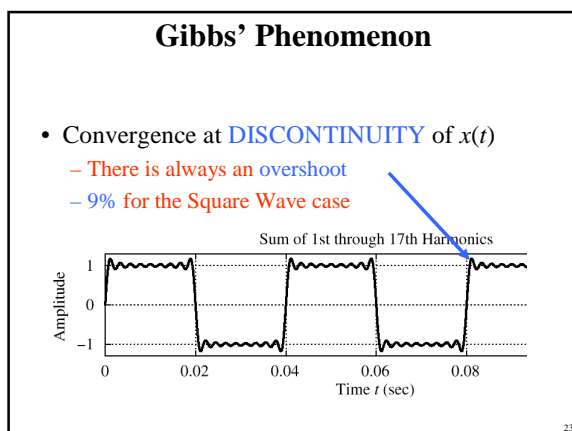
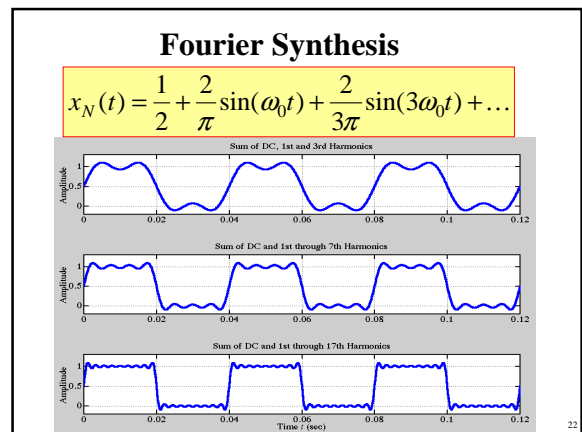
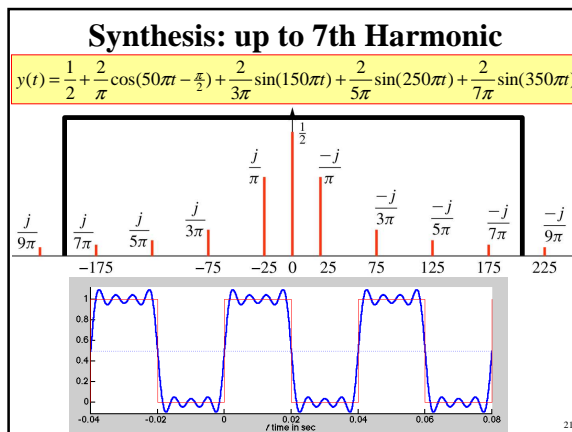
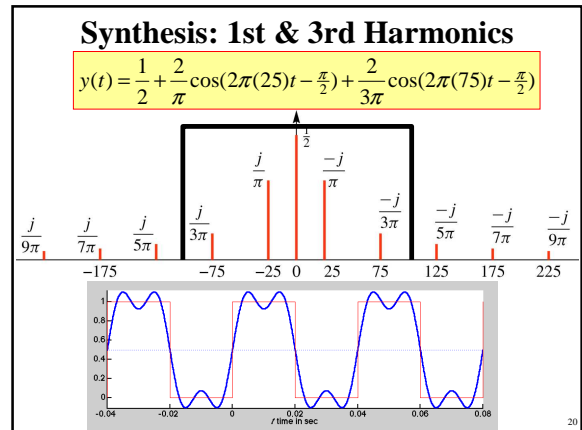
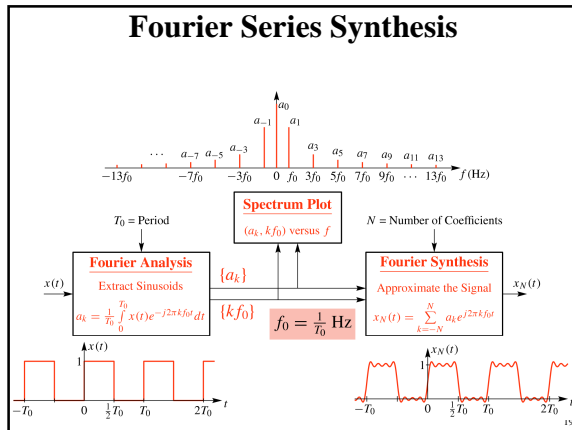
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^N a_k e^{j2\pi k f_0 t}$$

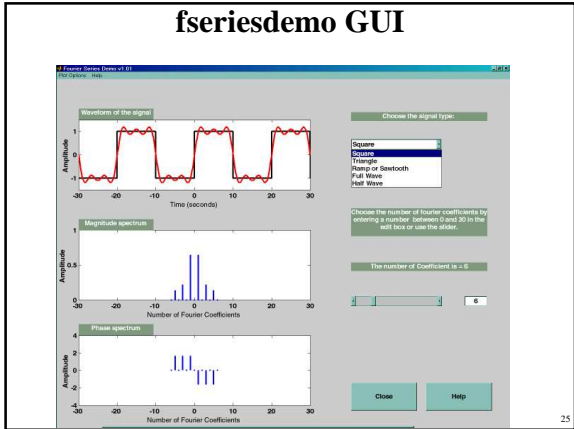
$a_{-k} = a_k^*$ when $x(t)$ is real

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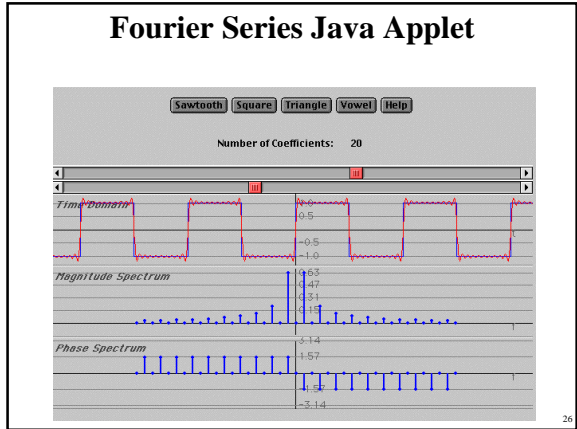


- ### Fourier Series Demos
- Fourier Series Java Applet
 - Greg Slabaugh
 - Interactive
 - http://users.ece.gatech.edu/mcclella/2025/Fsdemo_Slabaugh/fourier.html
 - MATLAB GUI: fseriesdemo
 - <http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html>

fseriesdemo GUI



Fourier Series Java Applet



Harmonic Signal (3 Freqs)

