Digital Signal Processing

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Digital Signal Processing

Lecture 2

Phase & Time-Shift

Complex Exponentials

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Sects. 2-3 to 2-5
- Appendix A: Complex Numbers
- Appendix B: MATLAB
- Next Lecture: finish Chap. 2,
  - Section 2-6 to end

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LECTURE OBJECTIVES

- Define Sinusoid Formula from a plot
- Relate TIME-SHIFT to PHASE

Introduce an **ABSTRACTION**:
Complex Numbers **represent** Sinusoids
Complex Exponential Signal

$$z(t) = Xe^{j\omega t}$$

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SINUSOIDAL SIGNAL

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** ω
  - **Radians/sec**
  - or, Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

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PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi / \omega = 2\pi / 0.3\pi = 20/3$$

- Determine a **peak** location by solving

$$(\omega t + \varphi) = 0$$

- **Peak at t=-4**

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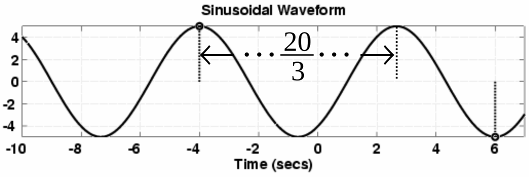
ANSWER for the PLOT

$$5\cos(0.3\pi t + 1.2\pi)$$

- Use T=20/3 and the peak location at t=-4



TIME-SHIFT

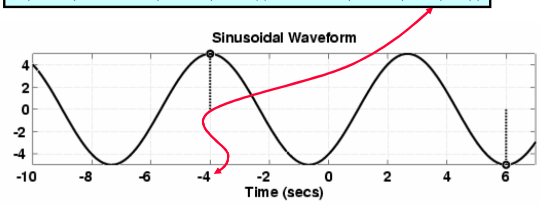
- In a mathematical formula we can replace t with $t-t_m$

$$x(t-t_m) = A\cos(\omega(t-t_m))$$

- Then the t=0 point moves to $t=t_m$
- Peak value of $\cos(\omega(t-t_m))$ is now at $t=t_m$

TIME-SHIFTED SINUSOID

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



PHASE <--> TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t + \varphi)$$

- and we obtain: $-\omega t_m = \varphi$
- or, $t_m = -\dfrac{\varphi}{\omega}$

TIME-SHIFT

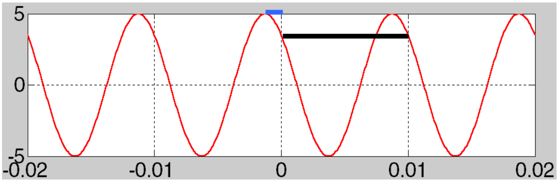
- Whenever a signal can be expressed in the form $x_1(t)=s(t-t_1)$, we say that $x_1(t)$ is time shifted version of $s(t)$
  - If t_1 is a + number, then the shift is to the right, and we say that the signal $s(t)$ has been *delayed* in time.
  - If t_1 is a - number, then the shift is to the left, and we say that the signal $s(t)$ was *advanced* in time.

SINUSOID from a PLOT

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
  - Compute phase: $\phi = -\omega\, t_m$
- Measure height of positive peak: A

3 steps

(A, ω, φ) from a PLOT



$$T = \frac{0.01\text{sec}}{1\text{ period}} = \frac{1}{100} \longrightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125\text{sec} \longrightarrow \varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

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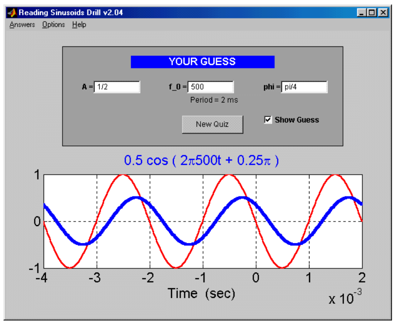
SINE DRILL (MATLAB GUI)



Sine drill

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PHASE is AMBIGUOUS

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t + \varphi + 2\pi) = A\cos(\omega t + \varphi)$$

  - Thus adding any multiple of 2π leaves *x(t)* unchanged

$$\text{if } t_m = \frac{-\varphi}{\omega}, \text{ then}$$

$$t_{m_2} = \frac{-(\varphi + 2\pi)}{\omega} = \frac{-\varphi}{\omega} - \frac{2\pi}{\omega} = t_m - T$$

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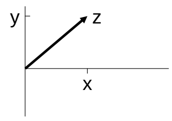
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - z = ***j***
  - Math and Physics use z = ***i***
- Complex number: z = x + ***j*** y



Cartesian coordinate system

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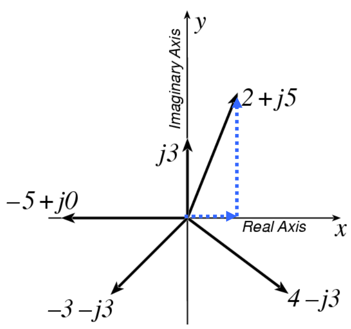
PLOT COMPLEX NUMBERS



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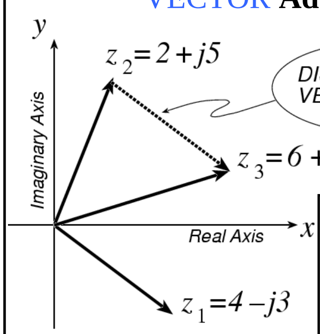
COMPLEX ADDITION = VECTOR Addition



$$\begin{aligned} z_3 &= z_1 + z_2 \\ &= (4 - j3) + (2 + j5) \\ &= (4+2) + j(-3+5) \\ &= 6 + j2 \end{aligned}$$

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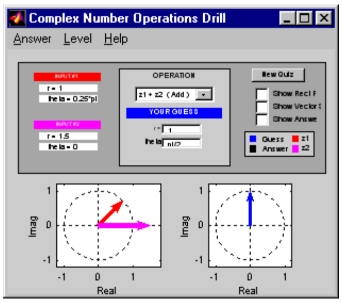
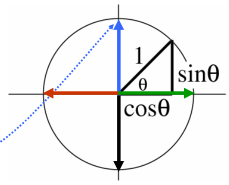


Complex drill

*** POLAR FORM ***

- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - **1** has angle of 0
  - **j** has angle of 0.5π
  - **–1** has angle of π
  - **– j** has angle of 1.5π
  - also, angle of **–j could** be –0.5π = 1.5π –2π
  - because the PHASE is **AMBIGUOUS**

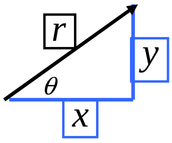


POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$



$$x = r\cos\theta$$

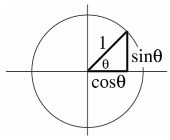
$$y = r\sin\theta$$

Most calculators do Polar-Rectangular

Need a notation for POLAR FORM

Euler's FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



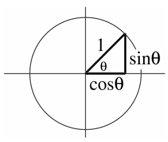
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - **θ = ωt**
  - Angle changes vs. time
  - ex: ω=20π rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Rotating phasors demo

cos = REAL PART

Real Part of Euler's $\cos(\omega t) = \Re e\{e^{j\omega t}\}$

General Sinusoid $x(t) = A\cos(\omega t + \varphi)$

So, $A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t + \varphi)}\}$

$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$

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REAL PART EXAMPLE

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate: $x(t) = \Re e\left\{-3je^{j\omega t}\right\}$

Answer:

$$x(t) = \Re e\left\{(-3j)e^{j\omega t}\right\}$$

$$= \Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\} = 3\cos(\omega t - 0.5\pi)$$

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COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

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