Digital Audio and Speech Processing

(Sayısal Ses ve Konuşma İşleme)

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IIR Filtering

LECTURE OBJECTIVES

- INTRODUCE FILTERING IDEA
  – Weighted Average
  – Running Average
- FINITE IMPULSE RESPONSE FILTERS
  – **FIR** Filters
- INFINITE IMPULSE RESPONSE FILTERS
  – **IIR** Filters
- Show how to compute the output $y[n]$ from the input signal, $x[n]$

DIGITAL FILTERING

- CONCENTRATE on the COMPUTER
  – PROCESSING ALGORITHMS
  – SOFTWARE (MATLAB)
  – HARDWARE: DSP chips, VLSI
- DSP: DIGITAL SIGNAL PROCESSING

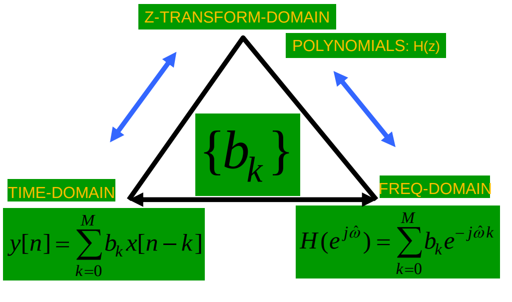
IIR Filters: Feedback and H(z)

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
  - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

  – Show how to compute the output $y[n]$
    - FIRST-ORDER CASE (N=1)
    - Z-transform: Impulse Response $h[n]$ ←→ $H(z)$

THREE DOMAINS



Quick Review: Delay by n_d

$$y[n] = x[n-n_d]$$

IMPULSE RESPONSE	$h[n] = \delta[n-n_d]$
SYSTEM FUNCTION	$H(z) = z^{-n_d}$
FREQUENCY RESPONSE	$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, $h[n]$
  - INFINITELY LONG
  - IIR Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

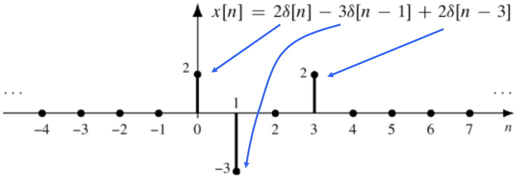
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB/OCTAVE
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$



COMPUTE *y[n]*

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8 y[n-1] + 5x[n]$$

- NEED y[-1] to get started

$$y[0] = 0.8 y[-1] + 5x[0]$$

AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

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COMPUTE MORE $y[n]$

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$
$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$
$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

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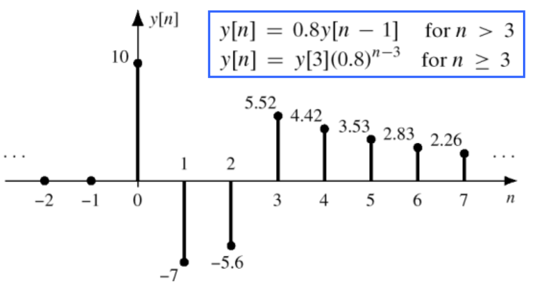
PLOT $y[n]$



$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$

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IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

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IMPULSE RESPONSE

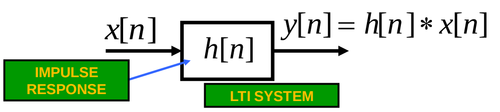
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

$$h[n] = 3(0.8)^n u[n]$$

- **CONVOLUTION** in TIME-DOMAIN



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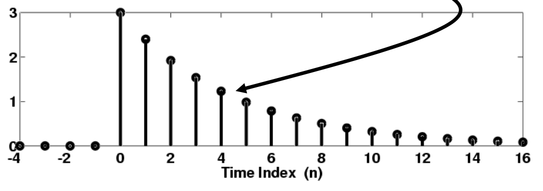
PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



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Infinite-Length Signal: *h*[*n*]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of *H*(*z*)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

H(*z*) = z-Transform{ *h*[*n*] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

H(*z*) = z-Transform{ *h*[*n*] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

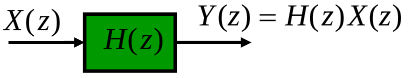
$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

z^{-1} is a shift

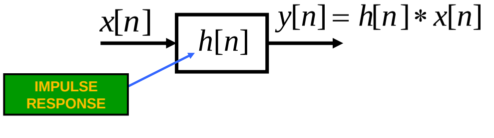
$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



STEP RESPONSE: *x*[*n*]=*u*[*n*]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

n	$x[n]$	$y[n]$ ($u[n]=1, \quad \text{for } n \geq 0$)
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
⋮	1	⋮

DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \quad \text{if } a_1 \neq 1$$

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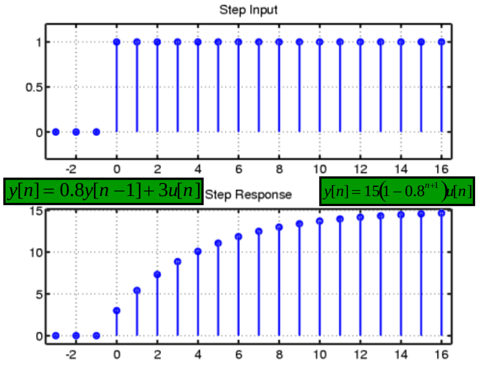
PLOT STEP RESPONSE



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IIR Filters: H(z) and Frequency Response

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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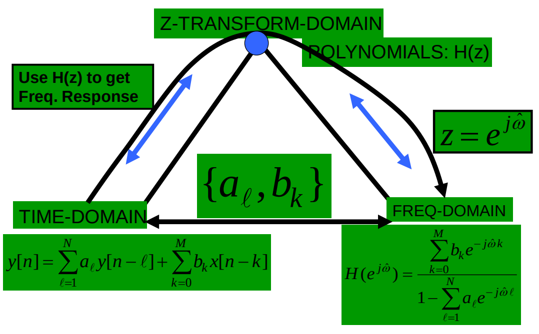
THREE DOMAINS



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$H(z)$ = z-Transform{ $h[n]$ }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

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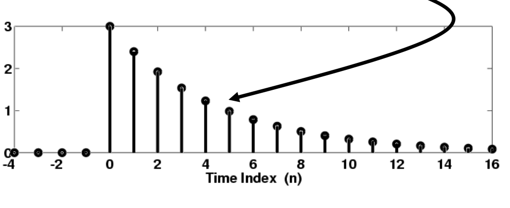
Typical IMPULSE Response

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



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First-Order Transform Pair

$$h[n] = ba^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

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DELAY PROPERTY of *X*(*z*)

- DELAY in TIME<-->Multiply $X(z)$ by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

Proof: $\displaystyle\sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

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Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

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SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

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SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

- **READ** the FILTER COEFFS: **H(z)**

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8 z^{-1}} \right) X(z)$$

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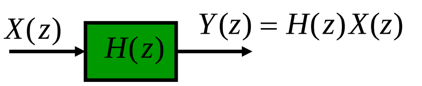
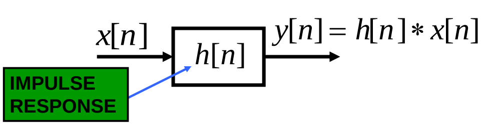
CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS



- **CONVOLUTION** in TIME-DOMAIN



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POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z)=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}} \rightarrow H(z)=\frac{b_0z+b_1}{z-a_1}$$

$$b_0z+b_1=0 \quad \Rightarrow z=-\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z-a_1=0 \quad \Rightarrow z=a_1$$

POLE: H(z) → inf

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EXAMPLE: Poles & Zeros

- VALUE of $H(z)$ at POLES is **INFINITE**

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

$$H(z)=\frac{2+2(-1)}{1-0.8(-1)}=0$$

ZERO at z= -1

$$H(z)=\frac{2+2(\frac{4}{5})^{-1}}{1-0.8(\frac{4}{5})^{-1}}=\frac{\frac{9}{2}}{0}\rightarrow\infty$$

POLE at z=0.8

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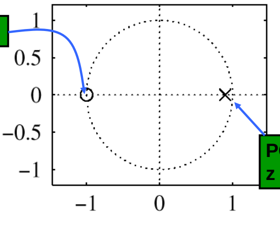
POLE-ZERO PLOT

ZERO at z = -1



POLE at z = 0.8

$$\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

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FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}})=H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n]\leftrightarrow H(z)\leftrightarrow H(e^{j\hat{\omega}})$$

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FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}})=H(z)\big|_{z=e^{j\hat{\omega}}}$$

$$H(z)=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}}$$

$$H(e^{j\hat{\omega}})=H(z)\big|_{z=e^{j\hat{\omega}}}=\frac{b_0+b_1e^{-j\hat{\omega}}}{1-a_1e^{-j\hat{\omega}}}$$

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FREQ. RESPONSE FORMULA

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}\rightarrow H(e^{j\hat{\omega}})=\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2=\left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2=\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\cdot\frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}}=\frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega}=0,\quad \left|H(e^{j\hat{\omega}})\right|^2=\frac{8+8}{0.04}=400,\quad @\,\hat{\omega}=\pi\,?$$

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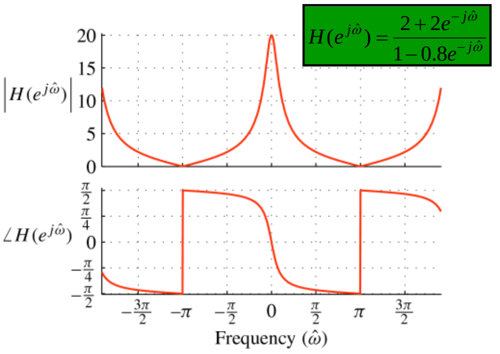
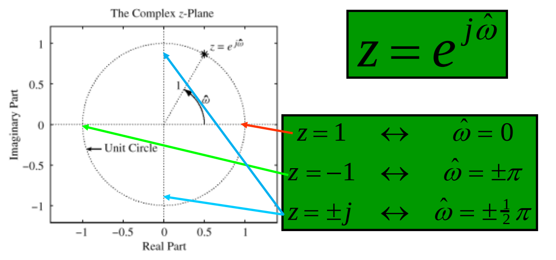
Frequency Response Plot



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

$$\text{if } \; x[n] = e^{j\hat{\omega}n}$$

$$\text{then } \; y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

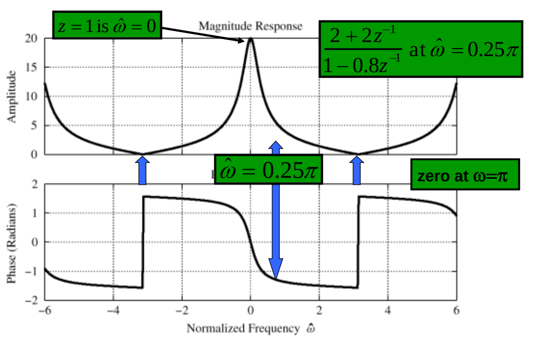
$$\text{where } \; H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

POP QUIZ

- Given: $H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
  - When

$$x[n] = \cos(0.25\pi n)$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

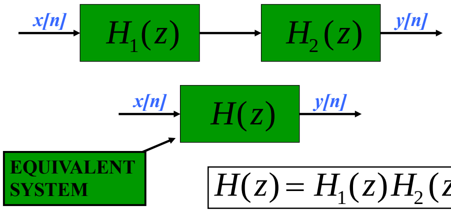
- Given: $H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find output, $y[n]$, when $x[n] = \cos(0.25\pi n)$
  - Evaluate at $z = e^{j0.25\pi}$

$$H(z) = \frac{2+2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

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