Digital Audio and Speech Processing

(Sayısal Ses ve Konuşma İşleme)

Prof. Dr. Nizamettin AYDIN

naydin@yildiz.edu.tr

nizamettinaydin@gmail.com

http://www3.yildiz.edu.tr/~naydin

Fourier Analysis

1

Fourier Series Coefficients

- Work with the Fourier Series Integral

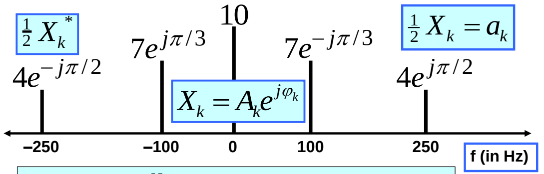
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: **$x(t+T_0) = x(t)$**
  - Spectrum from the Fourier Series

2

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

3

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

4

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

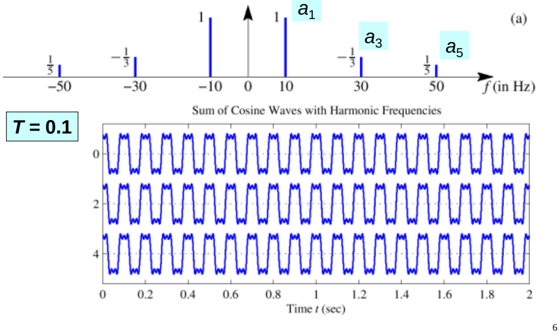
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

5

Harmonic Signal (3 Freqs)



6

SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

INTEGRAL Property of *exp(j)*

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m}e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m}-1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0$$

$$m \neq 0 \qquad \omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of *exp(j)*

- PRODUCT of *exp(+j)* and *exp(-j)*

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t}e^{-j(2\pi/T_0)kt}dt = \begin{cases}0 & k \neq \ell \\ 1 & k = \ell\end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \tfrac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}dt$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \sum_{k=-\infty}^{\infty} a_k\left(\tfrac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}dt\right) = a_\ell$$

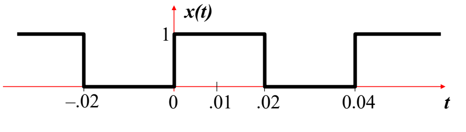
$$\Rightarrow a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Integral is zero except for $k = \ell$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases}1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0\end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

13

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

14

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

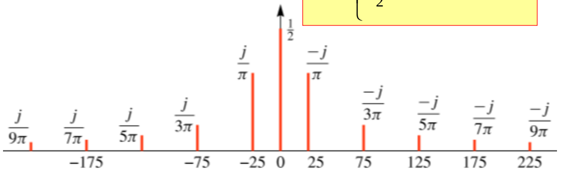
$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

15

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



16

Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Frequency $f_0 = 1/T_0$

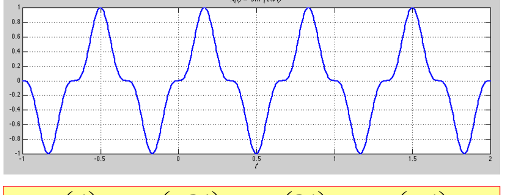
$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \tfrac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \text{(DC component)}$$

17

Fourier Series & Spectrum - Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

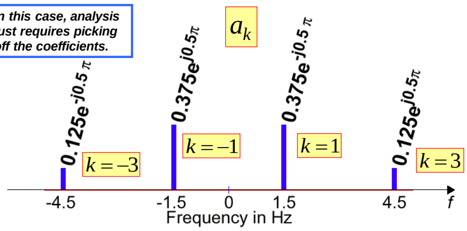
18

Fourier Series & Spectrum - Example

$x(t) = \sin^3(3\pi t)$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



19

STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

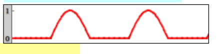
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

20

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \tfrac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\,e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j}e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

21

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\tfrac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \pm\frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$

22

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

23

Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

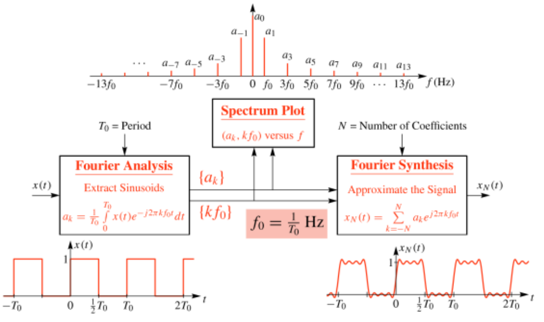
- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

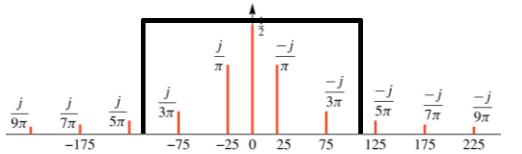
$a_{-k} = a_k^*$ when $x(t)$ is real

24

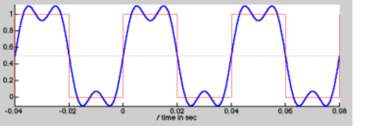
Fourier Series Synthesis



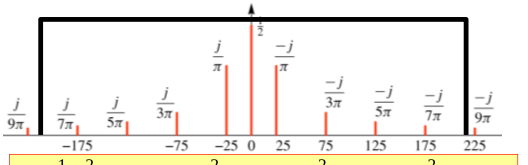
Synthesis: 1st & 3rd Harmonics



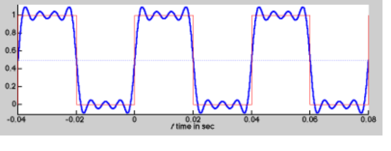
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$



Synthesis: up to 7th Harmonic

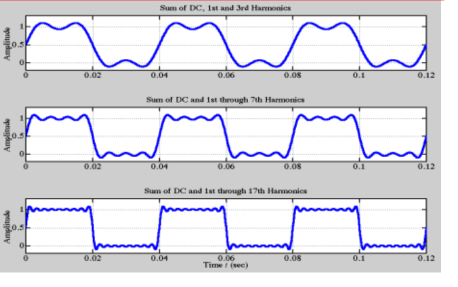


$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$



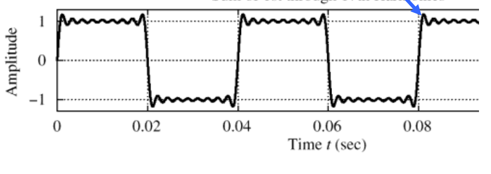
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of $x(t)$
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Transform

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Relation to Fourier Series

- Examples of Fourier transform pairs

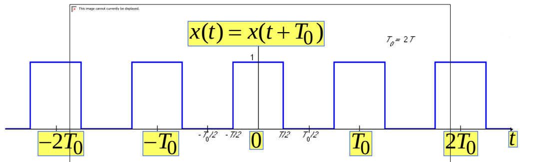
Everything = Sum of Sinusoid

- One Square Pulse = Sum of Sinusoids
  - ???????????
- Finite Length
- Not Periodic

- Limit of Square Wave as Period → infinity
  - Intuitive Argument

31

Fourier Series: Periodic *x(t)*



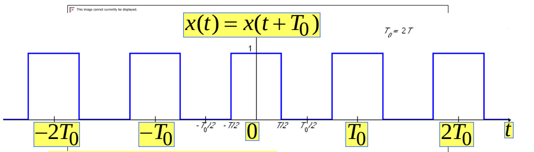
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$ ← ***Fourier Synthesis***

Fundamental Freq.
$\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t}\,dt$$ ← ***Fourier Analysis***

32

Square Wave Signal



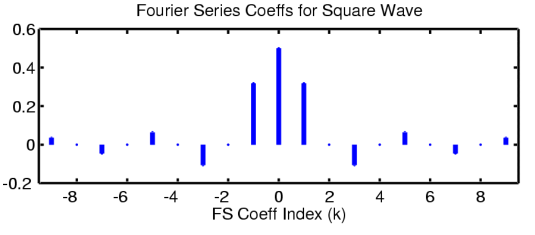
$$a_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t}\,dt$$

$$a_k = \left.\frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0}\right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

33

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



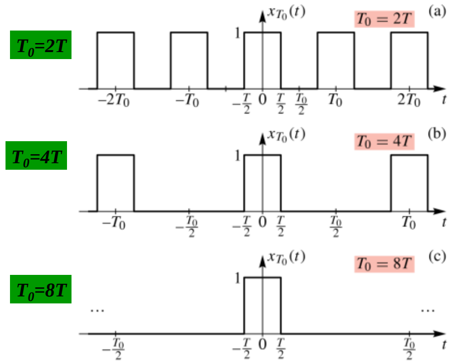
34

What if *x(t)* is not periodic?

- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t*.
- Fourier transform
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

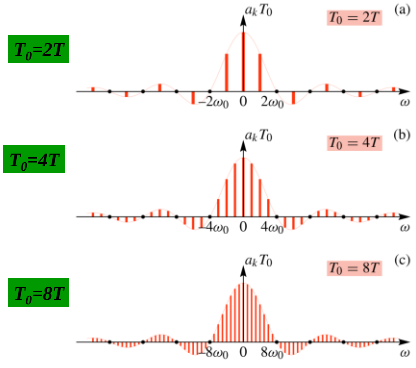
35

Limiting Behavior of FS



36

Limiting Behavior of Spectrum



Plot $(T_0 a_k)$

37

FS in the LIMIT (long period)

$$x_{T_0}(t) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\tfrac{2\pi}{T_0}\right) \mapsto x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega \qquad \lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

38

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

39

Example 1: $x(t) = e^{-at} u(t)$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$a > 0$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a + j\omega}\Bigg|_0^{\infty} = \frac{1}{a + j\omega}$$

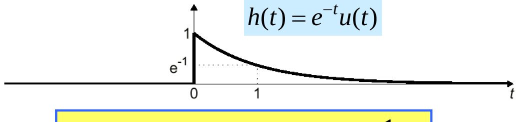
$$X(j\omega) = \frac{1}{a + j\omega}$$

40

Frequency Response

- Fourier Transform of $h(t)$ **is** the Frequency Response

$h(t) = e^{-t} u(t)$

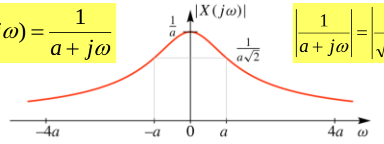


$$h(t) = e^{-t} u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

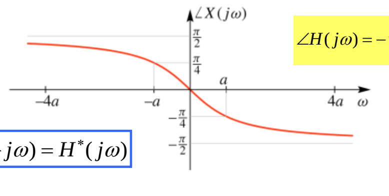
41

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

42

Example 2:

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases}$$

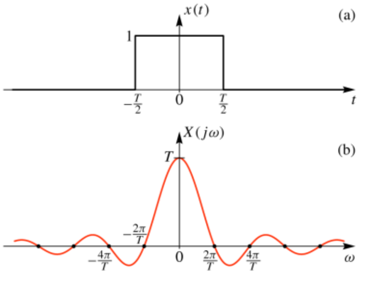
$$X(j\omega)=\int_{-T/2}^{T/2}(1)e^{-j\omega t}dt=\int_{-T/2}^{T/2}e^{-j\omega t}dt$$

$$X(j\omega)=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2}=\frac{e^{-j\omega T/2}-e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$

Example 2

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases}\Leftrightarrow X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3:

$$X(j\omega)=\begin{cases}1 & |\omega|<\omega_b\\ 0 & |\omega|>\omega_b\end{cases}$$

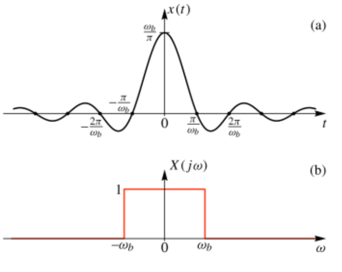
$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\omega_b}^{\omega_b}1e^{j\omega t}d\omega$$

$$x(t)=\frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_b}^{\omega_b}=\frac{1}{2\pi}\frac{e^{j\omega_b t}-e^{-j\omega_b t}}{jt}$$

$$x(t)=\frac{\sin(\omega_b t)}{\pi t}$$

Example 3

$$x(t)=\frac{\sin(\omega_b t)}{\pi t}\Leftrightarrow X(j\omega)=\begin{cases}1 & |\omega|<\omega_b\\ 0 & |\omega|>\omega_b\end{cases}$$



Example 4:

$$x(t)=\delta(t-t_0)$$

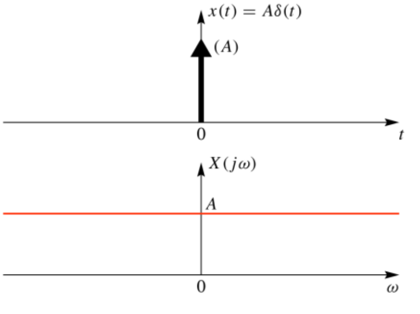
$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=1$$

Shifting Property of the Impulse

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t-t_0)e^{-j\omega t}dt=e^{-j\omega t_0}$$

Example 4

$$x(t)=\delta(t)\Leftrightarrow X(j\omega)=1$$



Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Example 5

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

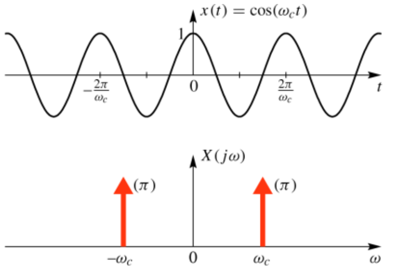


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform Properties

Fourier Transform - Properties

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis (Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$
$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- Manipulate the **"Frequency Spectrum"**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the "**Building Blocks**" ?
    - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers or Multipliers: $x(t)p(t)$

Fourier Transform of a General Periodic Signal

- If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$\text{Therefore, since } e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

55

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

56

FT of Complex Functions

- If $x(t)$ is a complex time function, i.e. $x(t)=x_r(t)+jx_i(t)$ where $x_r(t)$ and $x_i(t)$ are respectively the real part and imaginary part of the complex function $x(t)$,
- then the Fourier integral becomes

$$X(f) = \int_{-\infty}^{+\infty} [x_r(t) + jx_i(t)] e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} [x_r(t)\cos\omega t + x_i(t)\sin\omega t] dt$$

$$- j\int_{-\infty}^{+\infty} [x_r(t)\sin\omega t - x_i(t)\cos\omega t] dt = R(f) + jI(f)$$

$$F\{\cos\omega_0 t\} = \pi\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

$$F\{\sin\omega_0 t\} = \frac{\pi}{j}\{\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\}$$

57

Properties of the Fourier transform for complex time functions

Time domain ($x(t)$)	**Frequency domain ($X(f)$)**
Real	Real part even, imaginary part odd
Imaginary	Real part odd, imaginary part even
Real even, imaginary odd	Real
Real odd, imaginary even	Imaginary
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginary and odd	Real and odd
Complex and even	Complex and even
Complex and odd	Complex and odd

58

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

$$\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

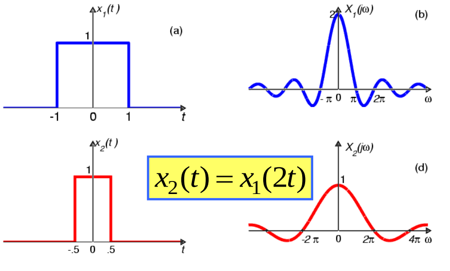
$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

$x(2t)$ shrinks; $\frac{1}{2} X(j\frac{\omega}{2})$ expands

59

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$



$$x_2(t) = x_1(2t)$$

60

Uncertainty Principle

- Try to make $x(t)$ shorter
  - Then $X(j\omega)$ will get wider
  - Narrow pulses have wide bandwidth
- Try to make $X(j\omega)$ narrower
  - Then $x(t)$ will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

61

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

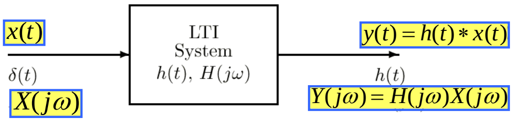
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

62

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to multiplication in the frequency-domain

$$Y(j\omega) = H(j\omega)X(j\omega)$$

63

Convolution Example

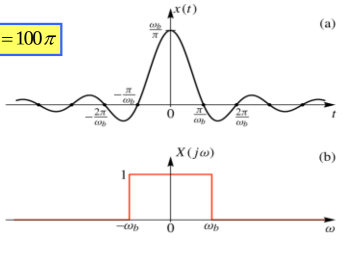
- Bandlimited **Input** Signal
  - "sinc" function
- Ideal LPF (Lowpass Filter)
  - $h(t)$ is a "sinc"
- **Output** is Bandlimited
  - Convolve "sincs"

64

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

$\omega_b = 100\pi$

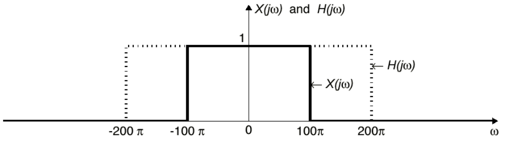


65

Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



66

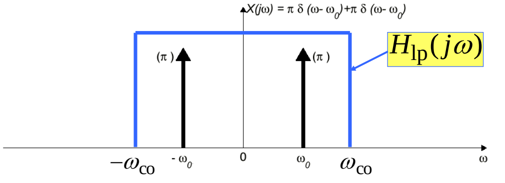
Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$= H(j\omega)[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$
$$= H(j\omega_0)\pi\delta(\omega-\omega_0)+H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

67

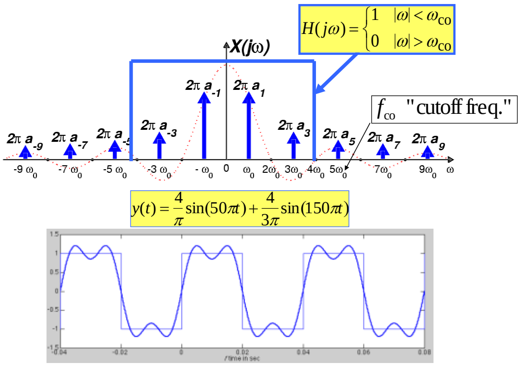
Ideal Lowpass Filter



$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$
$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

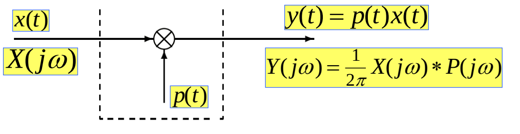
68

Ideal Lowpass Filter



69

Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

70

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

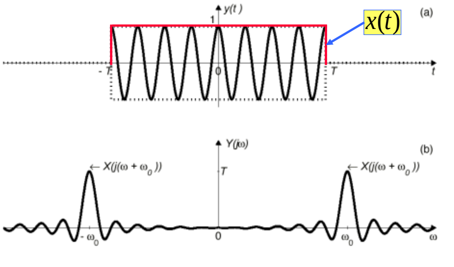
$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$
$$= X(j(\omega-\omega_0))$$

$$y(t)=\frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega)=\begin{cases}1 & \omega_0-7<\omega<\omega_0+7\\ 0 & elsewhere\end{cases}$$

71

Frequency Shifting Property

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$
$$Y(j\omega) = \tfrac{1}{2}X(j(\omega-\omega_0))+\tfrac{1}{2}X(j(\omega+\omega_0))$$



72

Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

73