Digital Audio and Speech Processing
(Sayısal Ses ve Konuşma İşleme)

Prof. Dr. Nizamettin AYDIN

naydin@yildiz.edu.tr
nizamettinaydin@gmail.com
http://www3.yildiz.edu.tr/~naydin

Fundamentals

Speech Processing

- Speech is the most natural form of human-human communications.
- Speech is related to language; linguistics is a branch of social science.
- Speech is related to human physiological capability; physiology is a branch of medical science.
- Speech is also related to sound and acoustics, a branch of physical science.
- Therefore, speech is one of the most intriguing signals that humans work with every day.

Speech Processing

- Purpose of speech processing:
  - To understand speech as a means of communication;
  - To represent speech for transmission and reproduction;
  - To analyze speech for automatic recognition and extraction of information
  - To discover some physiological characteristics of the talker.

Why Digital Processing of Speech?

- Digital processing of speech signals (DPSS) enjoys an extensive theoretical and experimental base developed over the past 75 years
- Much research has been done since 1965 on the use of digital signal processing in speech communication problems
- Highly advanced implementation technology (VLSI) exists that is well matched to the computational demands of DPSS
- There are abundant applications that are in widespread use commercially

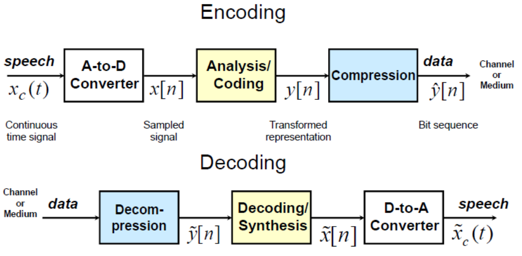
The Speech Stack

- Speech Applications
  - coding, synthesis, recognition, understanding, verification, language translation, speed-up/slow-down
- Speech Algorithms
  - speech-silence (background), voiced-unvoiced decision, pitch detection, formant estimation
- Speech Representations
  - temporal, spectral, homomorphic, LPC
- Fundamentals
  - acoustics, linguistics, pragmatics, speech perception

Speech Applications

- The top of the speech processing stack is applications
  - Speech coding
  - Speech synthesis
  - Speech recognition and understanding
  - Other speech applications

Speech Coding



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Speech Coding

- The process of transforming a speech signal into a representation for efficient transmission and storage of speech
  - Narrowband and broadband wired telephony
  - Cellular communications
  - Voice over IP (VoIP) to utilize the Internet as a real-time communications medium
  - Secure voice for privacy and encryption for national security applications
  - Extremely narrowband communications channels, e.g., battlefield applications using HF radio
  - Storage of speech for telephone answering machines, IVR systems, prerecorded messages

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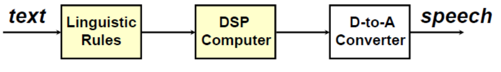
Speech Synthesis



- Process of generating a speech signal using computational means for effective human machine interactions
  - Machine reading of text or email messages
  - Telematics feedback in automobiles
  - Talking agents for automatic transactions
  - Automatic agent in customer care call center
  - Handheld devices such as foreign language phrasebooks, dictionaries, crossword puzzle helpers
  - Announcement machines that provide information such as stock quotes, airlines schedules, weather reports, etc.

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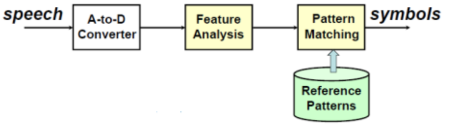
Pattern Matching Problems



- Speech recognition
- Speaker recognition
- Speaker verification
- Word spotting
- Automatic indexing of speech recordings

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Speech Recognition and Understanding

- The process of extracting usable linguistic information from a speech signal in support of human-machine communication by voice
  - Command and control (C&C) applications, e.g., simple commands for spreadsheets, presentation graphics, appliances
  - Voice dictation to create letters, memos, and other documents
  - Natural language voice dialogues with machines to enable Help desks, Call Centers
  - Voice dialing for cellphones and from PDA's and other small devices
  - Agent services such as calendar entry and update, address list modification and entry, etc.

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Other Speech Applications

- Speech Enhancement
  - for use in noisy environments, to eliminate echo, to align voices with video segments, to change voice qualities, to speed-up or slow-down prerecorded speech
    - potentially to improve intelligibility and naturalness of speech
- Speaker Verification
  - for secure access to premises, information, virtual spaces

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Other Speech Applications

- Speaker Recognition
  - for legal and forensic purposes
    - national security; also for personalized services
- Language Translation
  - to convert spoken words in one language to another to facilitate natural language dialogues between people speaking different languages,
    - i.e., tourists, business people
      - https://www.youtube.com/watch?v=WeByuOD8k1c

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DSP/Speech Enabled Devices

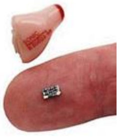
Internet Audio

Digital Cameras

PDAs & Streaming Audio/Video

Cell Phones

Hearing Aids



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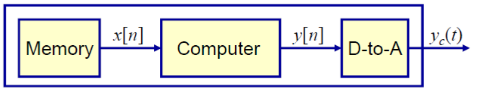
Apple iPod

- Stores music in MP3, AAC, MP4, wma, wav, … audio formats
- Compression of 11-to-1 for 128 kbps MP3
- Can store order of 20,000 songs with 30 GB disk
- Can use flash memory to eliminate all moving memory access
- Can load songs from iTunes store –
- Tens of millions sold



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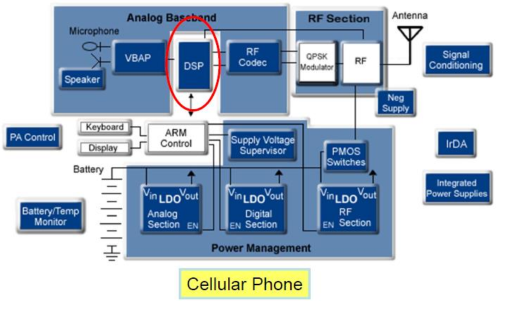
One of the Top DSP Applications



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Digital Speech Processing

- Need to understand the nature of the speech signal, and how DSP techniques, communication technologies, and information theory methods can be applied to help solve the various application scenarios
  - most of the course will concern itself with speech signal processing
    - i.e., converting one type of speech signal representation to another so as to uncover various mathematical or practical properties of the speech signal and do appropriate processing to aid in solving both fundamental and deep problems of interest

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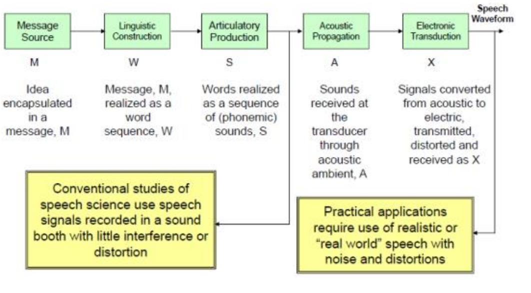
Speech Signal Production



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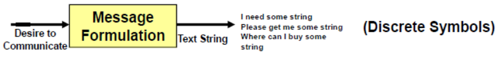
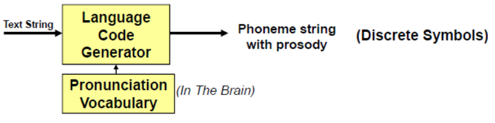
Speech Production/Generation Model

- Message Formulation → desire to communicate an idea, a wish, a request, … => express the message as a sequence of word

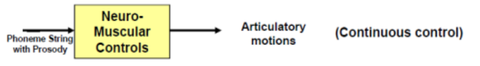


- Language Code → need to convert chosen text string to a sequence of sounds in the language that can be understood by others; need to give some form of emphasis, prosody (tune, melody) to the spoken sounds so as to impart non-speech information such as sense of urgency, importance, psychological state of talker, environmental factors (noise, echo)

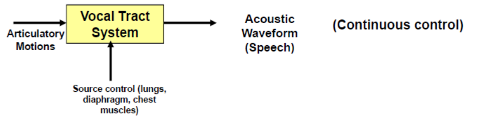


Speech Production/Generation Model

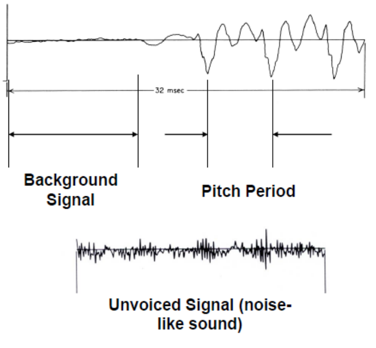
- Neuro-Muscular Controls → need to direct the neuro-muscular system to move the articulators (tongue, lips, teeth, jaws, velum) so as to produce the desired spoken message in the desired manner



- Vocal Tract System → need to shape the human vocal tract system and provide the appropriate sound sources to create an acoustic waveform (speech) that is understandable in the environment in which it is spoken

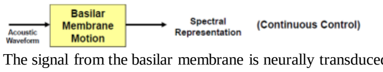


The Speech Signal

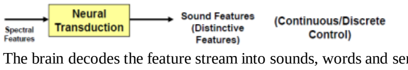


Speech Perception Model

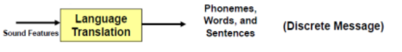
- The acoustic waveform impinges on the ear (the basilar membrane) and is spectrally analyzed by an equivalent filter bank of the ear



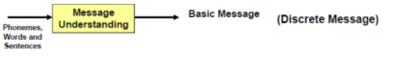
- The signal from the basilar membrane is neurally transduced and coded into features that can be decoded by the brain



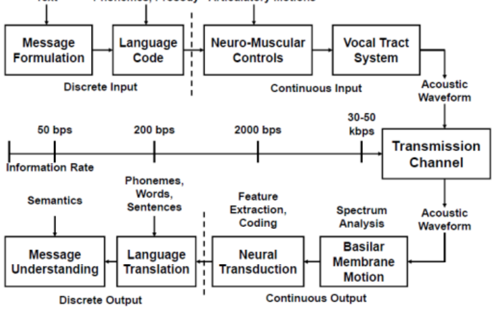
- The brain decodes the feature stream into sounds, words and sentences



- The brain determines the meaning of the words via a message understanding mechanism



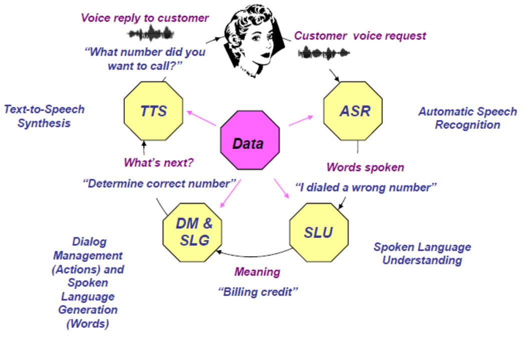
The Speech Chain



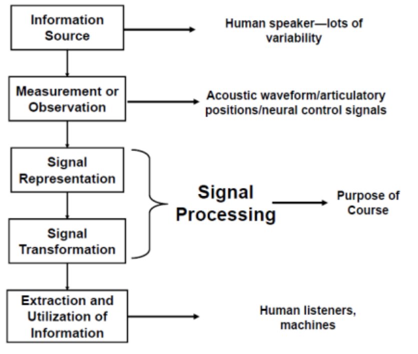
Speech Sciences

- Linguistics:
  - science of language, including phonetics, phonology, morphology, and syntax
- Phonemes:
  - smallest set of units considered to be the basic set of distinctive sounds of a languages (20-60 units for most languages)
- Phonemics:
  - study of phonemes and phonemic systems
- Phonetics:
  - study of speech sounds and their production, transmission, and reception, and their analysis, classification, and transcription
- Phonology:
  - phonetics and phonemics together
- Syntax:
  - meaning of an utterance

The Speech Circle



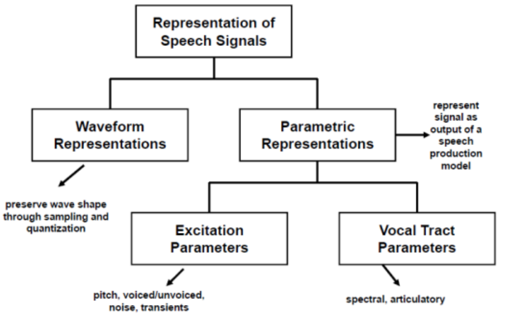
Signal Processing of Speech



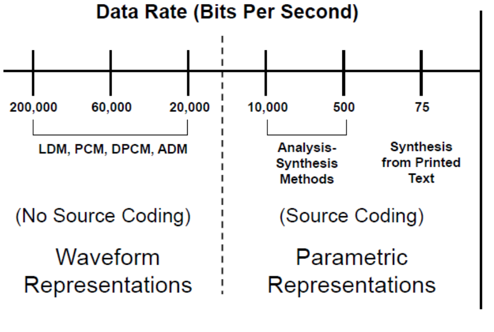
Digital Speech Processing

- DSP:
  - obtaining discrete representations of speech signal
  - theory, design and implementation of numerical procedures (algorithms) for processing the discrete representation in order to achieve a goal (recognizing the signal, modifying the time scale of the signal, removing background noise from the signal, etc.)
- Why DSP
  - reliability
  - flexibility
  - accuracy
  - real-time implementations on inexpensive dsp chips
  - ability to integrate with multimedia and data
  - encryptability/security of the data and the data representations via suitable techniques

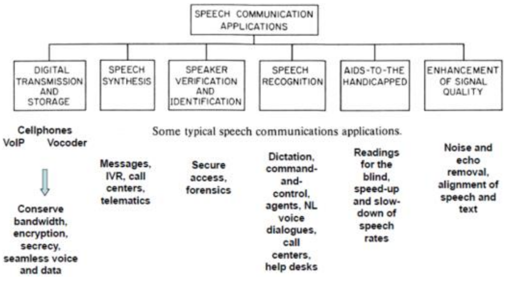
Hierarchy of Digital Speech Processing



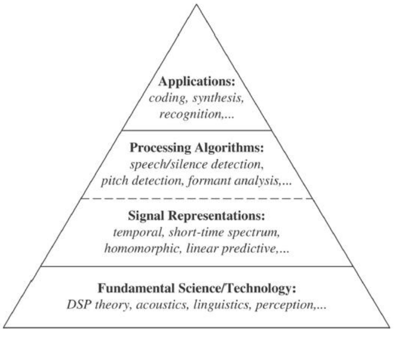
Information Rate of Speech



Speech Processing Applications



The Speech Stack



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Learning List

- Review some basic dsp concepts
- Speech production model
  - acoustics, articulatory concepts, speech production models
- Speech perception model
  - ear models, auditory signal processing, equivalent acoustic processing models
- Time domain processing concepts
  - speech properties, pitch, voiced-unvoiced, energy, autocorrelation, zero-crossing rates
- Short time Fourier analysis methods
  - digital filter banks, spectrograms, analysis-synthesis systems, vocoders

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Learning List

- Homomorphic speech processing
  - cepstrum, pitch detection, formant estimation, homomorphic vocoder
- Linear predictive coding methods
  - autocorrelation method, covariance method, lattice methods, relation to vocal tract models
- Speech waveform coding and source models
  - delta modulation, PCM, mu-law, ADPCM, vector quantization, multipulse coding, CELP coding
- Methods for speech synthesis and text-to-speech systems
  - physical models, formant models, articulatory models, concatenative models
- Methods for speech recognition
  - the Hidden Markov Model (HMM)

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Digital Signal Processing (DSP)

Dictionary definitions of the words in DSP:

- ***Digital***
  - ***operating by the use of discrete signals to represent data in the form of numbers***
- ***Signal***
  - ***a variable parameter by which information is conveyed through an electronic circuit***
- ***Processing***
  - ***to perform operations on data according to programmed instructions***
- **So a simple definition of DSP could be:**
  - **changing or analysing information which is measured as discrete sequences of numbers**
- **Unique features of DSP as opposed to ordinary digital processing:**
  - **signals come from the real world**
    - **this intimate connection with the real world leads to many unique needs such as the need to react in real time and a need to measure signals and convert them to digital numbers**
  - **signals are discrete**
    - **which means the information in between discrete samples is lost**

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WHY USE DSP ?

- **Versatility:**
  - digital systems can be reprogrammed for other applications
  - digital systems can be ported to different hardware
- **Repeatability:**
  - digital systems can be easily duplicated
  - digital systems do not depend on strict component tolerances
  - digital system responses do not drift with temperature
- **Simplicity:**
  - some things can be done more easily digitally than with analogue systems

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DSP is used in a very wide variety of applications

- Radar, sonar, telephony, audio, multimedia, communications, ultrasound, process control, digital camera, digital tv, Telecommunications, Sound & Music, Fourier Optics, X-ray Crystallography, Protein Structure & DNA, Computerized Tomography, Nuclear Magnetic Resonance: MRI,Radioastronomy
- All these applications share some common features:
  - they use a lot of maths (multiplying and adding signals)
  - they deal with signals that come from the real world
  - they require a response in a certain time
- Where general purpose DSP processors are concerned, most applications deal with signal frequencies that are in the audio range

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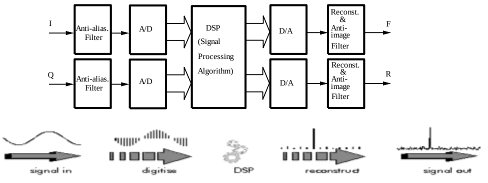
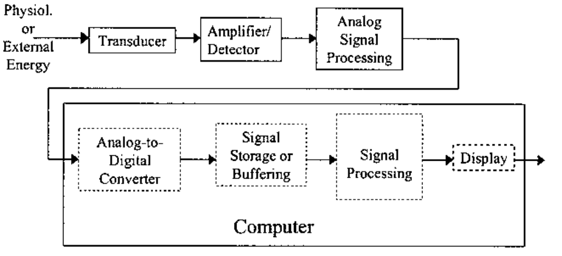
Fundamental concepts in DSP

- DSP applications deal with analogue signals
  - the analogue signal has to be converted to digital form

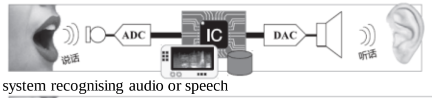


A typical measurement system

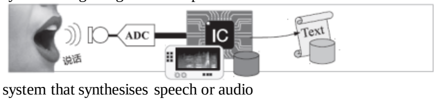


Three classes of digital audio system

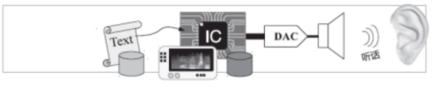
- A complete digital audio processing path
  - incuding an input microphone, amplifier, ADC, processing system, DAC, amplifier and loudspeaker



- A system recognising audio or speech



- A system that synthesises speech or audio



Transducers

- A “transducer” is a device that converts energy from one form to another.
- In signal processing applications, the purpose of energy conversion is to transfer information, not to transform energy.
- In physiological measurement systems, transducers may be
  - input transducers (or sensors)
    - they convert a non-electrical energy into an electrical signal.
    - for example, a microphone.
  - output transducers (or actuators)
    - they convert an electrical signal into a non-electrical energy.
    - For example, a speaker.

Analog - Digital

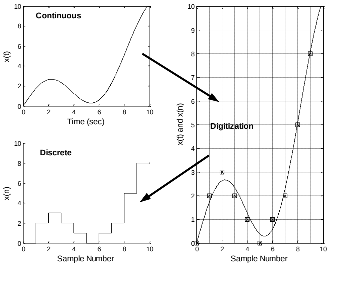
- The analog signal
  - a continuous variable defined with infinite precision

  is converted to a discrete sequence of measured values which are represented digitally
- Information is lost in converting from analog to digital, due to:
  - inaccuracies in the measurement
  - uncertainty in timing
  - limits on the duration of the measurement
- These effects are called quantisation errors

Signal Encoding: Analog-to Digital Conversion

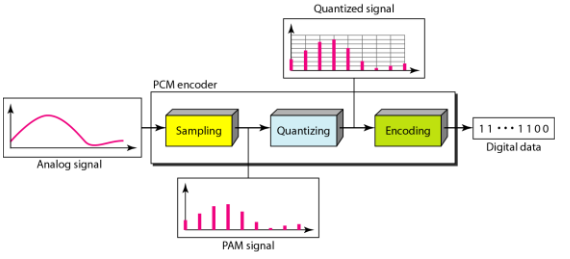
Continuous (analog) signal ↔ Discrete signal

x(t) = f(t) ↔ Analog to digital conversion ↔ *x(n) = x(1), x(2), x(3), ... x(n)*



Analog-to Digital Conversion

- ADC consists of four steps to digitize an analog signal:
  1. Filtering
  2. Sampling
  3. Quantization
  4. Binary encoding
- Before we sample, we have to filter the signal to limit the maximum frequency of the signal as it affects the sampling rate.
- Filtering should ensure that we do not distort the signal, ie remove high frequency components that affect the signal shape.

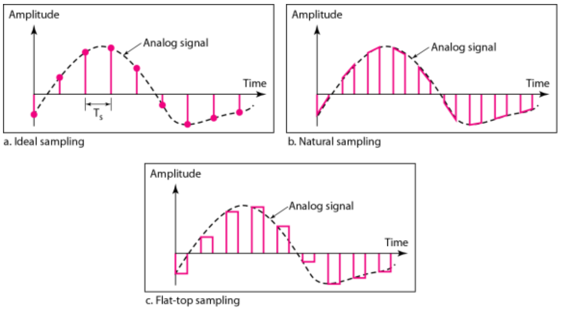


Sampling

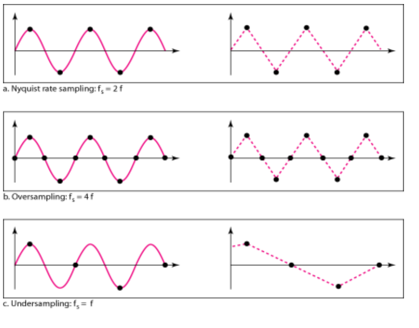
- The sampling results in a discrete set of digital numbers that represent measurements of the signal
  - usually taken at equal intervals of time
- Sampling takes place after the hold
  - The hold circuit must be fast enough that the signal is not changing during the time the circuit is acquiring the signal value
- We don't know what we don't measure
- In the process of measuring the signal, some information is lost

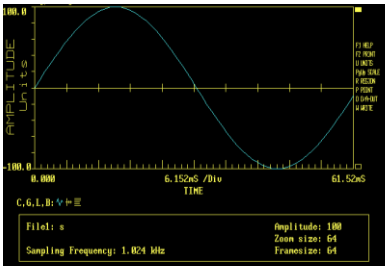
Sampling

- Analog signal is sampled every T_S secs.
- T_s is referred to as the sampling interval.
- $f_s = 1/T_s$ is called the sampling rate or sampling frequency.
- There are 3 sampling methods:
  - Ideal - an impulse at each sampling instant
  - Natural - a pulse of short width with varying amplitude
  - Flattop - sample and hold, like natural but with single amplitude value
- The process is referred to as pulse amplitude modulation PAM and the outcome is a signal with analog (non integer) values



Recovery of a sampled sine wave for different sampling rates





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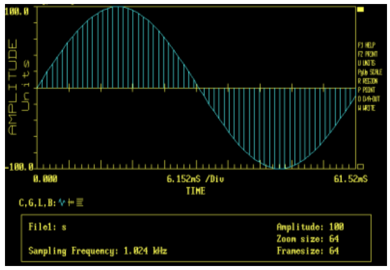

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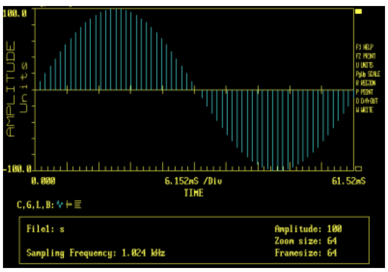

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Storing Digital Signal

- ***x*[*n*]** is a sampled signal
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
- Time between samples is
  - 1/44100 = 22.67 microsec
- 16-bit samples
- Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2*(16/8)*60*44100 = 10584000 = 10.584 Mbytes

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Sampling Theorem

$$F_s \geq 2f_m$$

According to the Nyquist theorem, the sampling rate must be at least 2 times the highest frequency contained in the signal.

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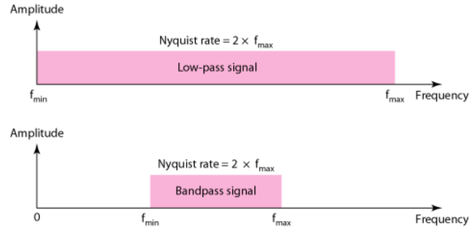
Nyquist sampling rate for low-pass and bandpass signals



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Quantization

- Sampling results in a series of pulses of varying amplitude values ranging between two limits:
  - a min and a max.
- The amplitude values are infinite between the two limits.
- We need to map the infinite amplitude values onto a finite set of known values.
- This is achieved by dividing the distance between min and max into L zones, each of height Δ.

$$\Delta = (\max - \min)/L$$

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Quantization Levels

- The midpoint of each zone is assigned a value from 0 to L-1 (resulting in L values)
- Each sample falling in a zone is then approximated to the value of the midpoint.

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Quantization Zones

- Assume we have a voltage signal with amplitutes
  - V_{min}= -20V and V_{max}= +20V.
- We want to use L=8 quantization levels.
- Zone width $\Delta = (20 - (-20))/8 = 5$
- The 8 zones are:
  - -20 to -15, -15 to -10, -10 to -5, -5 to 0, 0 to +5, +5 to +10, +10 to +15, +15 to +20
- The midpoints are:
  - -17.5, -12.5, -7.5, -2.5, 2.5, 7.5, 12.5, 17.5

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Assigning Codes to Zones

- Each zone is then assigned a binary code.
- The number of bits required to encode the zones, or the number of bits per sample as it is commonly referred to, is obtained as follows:

$$n_b = \log_2 L$$

- Given our example, $n_b = 3$
- The 8 zone (or level) codes are therefore:
  - 000, 001, 010, 011, 100, 101, 110, and 111
- Assigning codes to zones:
  - 000 will refer to zone -20 to -15
  - 001 to zone -15 to -10, etc.

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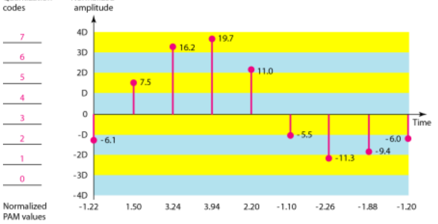
Quantization and encoding of a sampled signal



Normalized PAM values	-1.22	1.50	3.24	3.94	2.20	-1.10	-2.26	-1.88	-1.20
Normalized quantized values	-1.50	1.50	3.50	3.50	2.50	-1.50	-2.50	-1.50	-1.50
Normalized error	-0.38	0	+0.26	-0.44	+0.30	-0.40	-0.24	+0.38	-0.30
Quantization code	2	5	7	7	6	2	1	2	2
Encoded words	010	101	111	111	110	010	001	010	010

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Quantization Error

- When a signal is quantized, we introduce an error
  - the coded signal is an approximation of the actual amplitude value.
- The difference between actual and coded value (midpoint) is referred to as the quantization error.
- The more zones, the smaller Δ
  - which results in smaller errors.
- BUT, the more zones the more bits required to encode the samples
  - higher bit rate

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Analog-to-digital Conversion

Example An 12-bit analog-to-digital converter (ADC) advertises an accuracy of ± the least significant bit (LSB). If the input range of the ADC is 0 to 10 volts, what is the accuracy of the ADC in analog volts?

Solution: If the input range is 10 volts then the analog voltage represented by the LSB would be:

$$V_{LSB} = \frac{V_{max}}{2^{\text{Nu bits}}} = \frac{10}{2^{12}} = \frac{10}{4096} = .0024 \text{ volts}$$

Hence the accuracy would be ± .0024 volts.

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Sampling related concepts

- Over/exact/under sampling
- Regular/irregular sampling
- Linear/Logarithmic sampling
- Aliasing
- Anti-aliasing filter
- Image
- Anti-image filter

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Steps for digitization/reconstruction of a signal

- Band limiting (LPF)
- Sampling / Holding
- Quantization
- Coding

These are basic steps for A/D conversion

- D/A converter
- Sampling / Holding
- Image rejection

These are basic steps for reconstructing a sampled digital signal

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Digital data: end product of A/D conversion and related concepts

- Bit: least digital information, binary 1 or 0
- Nibble: 4 bits
- Byte: 8 bits, 2 nibbles
- Word: 16 bits, 2 bytes, 4 nibbles
- Some jargon:
  – integer, signed integer, long integer, 2s complement, hexadecimal, octal, floating point, etc.

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Data types

- Our first requirement is to find a way to represent information (data) in a form that is mutually comprehensible by human and machine.
  – Ultimately, we will have to develop schemes for representing all conceivable types of information - language, images, actions, etc.
  – We will start by examining different ways of representing integers, and look for a form that suits the computer.
  – Specifically, the devices that make up a computer are switches that can be on or off, i.e. at high or low voltage.
    - Thus they naturally provide us with two symbols to work with:
      – we can call them on & off, or (more usefully) 0 and 1.

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Signal

- An information variable represented by physical quantity.
- For digital systems, the variable takes on discrete values.
- Two level, or binary values are the most prevalent values in digital systems.
- Binary values are represented abstractly by:
  - digits 0 and 1
  - words (symbols) False (F) and True (T)
  - words (symbols) Low (L) and High (H)
  - and words On and Off.
- Binary values are represented by values or ranges of values of physical quantities

Number Systems – Representation

- Positive radix, positional number systems
- A number with *radix* ***r*** is represented by a string of digits:

$$A_{n-1}A_{n-2}\ldots A_1A_0\,.\,A_{-1}A_{-2}\ldots A_{-m+1}A_{-m}$$

in which $0 \le A_i < r$ and **.** is the radix point.

- The string of digits represents the power series:

$$(\text{Number})_r = \left(\sum_{i=0}^{i=n-1} A_i \cdot r^i\right) + \left(\sum_{j=-m}^{j=-1} A_j \cdot r^j\right)$$

(Integer Portion) + (Fraction Portion)

Decimal Numbers

- "decimal" means that we have ten digits to use in our representation (the symbols 0 through 9)
- What is 3546?
  - it is *three* thousands plus *five* hundreds plus *four* tens plus *six* ones.
  - i.e. $3546 = 3.10^3 + 5.10^2 + 4.10^1 + 6.10^0$
- How about negative numbers?
  - we use two more symbols to distinguish positive and negative:

  + and -

Unsigned Binary Integers

$Y = \text{"abc"} = a.2^2 + b.2^1 + c.2^0$

(where the digits a, b, c can each take on the values of 0 or 1 only)

N = number of bits

Range is:
$0 \le i < 2^N - 1$

	3-bits	5-bits	8-bits
0	000	00000	00000000
1	001	00001	00000001
2	010	00010	00000010
3	011	00011	00000011
4	100	00100	00000100

Problem:

- How do we represent *negative* numbers?

Two's Complement

- Transformation
  - To transform a into -a, invert all bits in a and add 1 to the result

Range is:
$-2^{N-1} < i < 2^{N-1} - 1$

Advantages:

- Operations need not check the sign
- Only one representation for zero
- Efficient use of all the bits

-16	10000
…	…
-3	11101
-2	11110
-1	11111
0	00000
+1	00001
+2	00010
+3	00011
…	…
+15	01111

Limitations of integer representations

- Most numbers are not integer!
  - Even with integers, there are two other considerations:
- Range:
  - The magnitude of the numbers we can represent is determined by how many bits we use:
    - e.g. with 32 bits the largest number we can represent is about +/- 2 billion, far too small for many purposes.
- Precision:
  - The exactness with which we can specify a number:
    - e.g. a 32 bit number gives us 31 bits of precision, or roughly 9 figure precision in decimal repesentation.
- We need another data type!

Real numbers

- Our decimal system handles non-integer *real* numbers by adding yet another symbol - the decimal point (**.**) to make a *fixed point* notation:
  - e.g. 3456.78 = $3.10^3 + 4.10^2 + 5.10^1 + 6.10^0 + 7.10^{-1} + 8.10^{-2}$
- The *floating point*, or scientific, notation allows us to represent very large and very small numbers (integer or real), with as much or as little precision as needed:
  - Unit of electric charge e = $1.602\ 176\ 462 \times 10^{-19}$ Coulomb
  - Volume of universe = 1×10^{85} cm^3
    - the two components of these numbers are called the mantissa and the exponent

Real numbers in binary

- We mimic the decimal floating point notation to create a "hybrid" binary floating point number:
  - We first use a "binary point" to separate whole numbers from fractional numbers to make a fixed point notation:
    - e.g. 00011001.110 = $1.2^4 + 1.2^3 + 1.2^1 + 1.2^{-1} + 1.2^{-2}$ => 25.75
      (2^{-1} = 0.5 and 2^{-2} = 0.25, etc.)
  - We then "float" the binary point:
    - 00011001.110 => 1.1001110 x 2^4
      mantissa = 1.1001110, exponent = 4
  - Now we have to express this without the extra symbols (x, 2, .)
    - by convention, we divide the available bits into three fields:
      sign, mantissa, exponent

IEEE-754 fp numbers - 1

s	biased exp.	fraction
1	8 bits	23 bits

32 bits:

$N = (-1)^s \times 1.\mathbf{fraction} \times 2^{(\text{biased exp.} - 127)}$

- Sign: 1 bit
- Mantissa: 23 bits
  - We "normalize" the mantissa by dropping the leading 1 and recording only its fractional part (why?)
- Exponent: 8 bits
  - In order to handle both +ve and -ve exponents, we add 127 to the actual exponent to create a "biased exponent":
    - 2^{-127} => biased exponent = 0000 0000 (= 0)
    - 2^0 => biased exponent = 0111 1111 (= 127)
    - 2^{+127} => biased exponent = 1111 1110 (= 254)

IEEE-754 fp numbers - 2

- Example: Find the corresponding fp representation of 25.75
  - 25.75 => 00011001.110 => 1.1001110 x 2^4
  - sign bit = 0 (+ve)
  - normalized mantissa (fraction) = 100 1110 0000 0000 0000 0000
  - biased exponent = 4 + 127 = 131 => 1000 0011
  - so 25.75 => 0 1000 0011 100 1110 0000 0000 0000 0000 => x41CE0000
- Values represented by convention:
  - Infinity (+ and -): exponent = 255 (1111 1111) and fraction = 0
  - NaN (not a number): exponent = 255 and fraction ≠ 0
  - Zero (0): exponent = 0 and fraction = 0
    - note: exponent = 0 => fraction is *de-normalized*, i.e no hidden 1

Binary Numbers and Binary Coding

- **Flexibility of representation**
  - **Within constraints below, can assign any binary combination (called a code word) to any data as long as data is uniquely encoded.**
- **Information Types**
  - **Numeric**
    - **Must represent range of data needed**
    - **Very desirable to represent data such that simple, straightforward computation for common arithmetic operations permitted**
    - **Tight relation to binary numbers**
  - **Non-numeric**
    - **Greater flexibility since arithmetic operations not applied.**
    - **Not tied to binary numbers**

Non-numeric Binary Codes

- Given *n* binary digits (called bits), a binary code is a mapping from a set of represented elements to a subset of the 2^n binary numbers.
- Example:
  - A binary code for the seven colors of the rainbow
    - Code 100 is not used

Color	Binary Number
Red	000
Orange	001
Yellow	010
Green	011
Blue	101
Indigo	110
Violet	111

Number of Bits Required

- Given M elements to be represented by a binary code, the minimum number of bits, n, needed, satisfies the following relationships:
  $2^n > M > 2^{(n-1)}$
  $n = \lceil \log_2 M \rceil$ where $\lceil x \rceil$, called the *ceiling function,* is the integer greater than or equal to x.
- **Example:**
- How many bits are required to represent decimal digits with a binary code?
  – 4 bits are required ($n = \lceil \log_2 9 \rceil = 4$)

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