Digital Signal Processing

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Course Details

- Course Code : SEN522
- Course Name: Digital Signal Processing
  (Sayısal İşaret İşleme)
- Instructor : Nizamettin AYDIN

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Example 1

Use one of the following trigonometric identities to derive an expression for **cos8θ** in terms of **cos9θ**, **cos7θ**, and **cosθ**.

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2\sin\theta\cos\theta$
4	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

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$\cos 9\theta = \cos(8\theta + \theta) = \cos 8\theta \cos\theta - \sin 8\theta \sin\theta$

$\cos 7\theta = \cos(8\theta - \theta) = \cos 8\theta \cos\theta + \sin 8\theta \sin\theta$

$\cos 9\theta + \cos 7\theta = 2\cos\theta\cos 8\theta$

$\Rightarrow \cos 8\theta = \dfrac{\cos 9\theta + \cos 7\theta}{2\cos\theta}$

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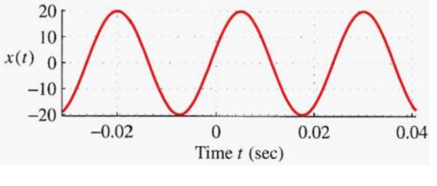
Example 2

- In the following figure, it is possible to measure both a positive and a negative value of t_1 and then calculate the corresponding phase shifts.

  Which phase shift is within the range $-\pi < Ø \leq \pi$?
  Verify that the two phase shifts differ by 2π.



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Positive t_1 is $t_1 = 0.005$ sec

$\varphi = -\omega_0 t_1 = -2\pi(40)(0.005) = -2\pi(0.2)$

Negative t_1 is $t_1 = -0.02$ sec

$\varphi = -\omega_0 t_1 = -2\pi(40)(-0.02) = 2\pi(0.8)$

Difference $= 2\pi(0.8) - (-2\pi(0.2)) = 2\pi$ ✓

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P-4.1 Consider the cosine wave

$$x(t) = 10\cos(880\pi t + \phi)$$

Suppose that we obtain a sequence of numbers by sampling the waveform at equally spaced time instants nT_s. In this case, the resulting sequence would have values

$$x[n] = x(nT_s) = 10\cos(880\pi n T_s + \phi)$$

for $-\infty < n < \infty$. Suppose that $T_s = 0.0001$ sec.

(a) How many samples will be taken in one period of the cosine wave?

(b) Now consider another waveform $y(t)$ such that

$$y(t) = 10\cos(\omega_0 t + \phi)$$

Find a frequency $\omega_0 > 880\pi$ such that $y(nT_s) = x(nT_s)$ for all integers n.

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(a) $x[n] = x(nT_s) = 10\cos(880\pi n T_s + \varphi)$ $\quad T_s = 0.0001$

$880 T_s = 880 \times 10^{-4} = 0.088 = 11/125$

To find the number of samples within one period of the continuous cosine $x(t)$, find the largest integer satisfying $880\pi n T_s \le 2\pi$

$n \le \frac{2}{0.088} = \frac{250}{11} = 22.73$

There are 23 samples in one period, because samples $n = 0, 1, 2, \dots 22$ are within one period.

NOTE: the period of $x[n]$ is <u>not</u> 23; it is actually 250.

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(b) $y[n] = 10\cos(\omega_0 n T_s + \varphi)$

To get the same samples for $x[n]$ & $y[n]$ we solve:

$\omega_0 n T_s = 880\pi n T_s + 2\pi \ell n$ $\quad \ell = \text{integer}$

$\Rightarrow \omega_0 = 880\pi + \frac{2\pi\ell}{T_s}$ $\quad \frac{2\pi}{T_s} = 20{,}000\pi$

Take $\ell = 1$: $\omega_0 = 20{,}880\pi$

(c) Find largest integer satisfying

$(20{,}880\pi) n T_s \le 2\pi$

$n \le \frac{2}{2.088}$ which is less than one!

$\therefore$ only one sample per period is taken

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P-4.2 Let $x(t) = 7\sin(11\pi t)$. In each of the following parts, the discrete-time signal $x[n]$ is obtained by sampling $x(t)$ at a rate f_s, and the resultant $x[n]$ can be written as

$$x[n] = A\cos(\hat{\omega}_0 n + \phi)$$

For each part below, determine the values of A, ϕ, and $\hat{\omega}_0$. In addition, state whether or not the signal has been over-sampled or under-sampled.

(a) Sampling frequency is $f_s = 10$ samples/sec.

(b) Sampling frequency is $f_s = 5$ samples/sec.

(c) Sampling frequency is $f_s = 15$ samples/sec.

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$x(t) = 7\sin(11\pi t) = 7\cos(11\pi t - \pi/2) \xrightarrow{} \boxed{A/D}_{f_s} \xrightarrow{} x[n] = A\cos(\hat{\omega}_0 n + \varphi)$.

(a) $f_s = 10$ samples/sec.

$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{f_s}\right) = 7\cos\left(\frac{11\pi n}{10} - \pi/2\right)$.

$= 7\cos\left(\frac{11\pi n}{10} - 2\pi n - \pi/2\right)$.

$= 7\cos\left(-\frac{9\pi n}{10} - \pi/2\right) = 7\cos(.9\pi n + \pi/2)$.

$\boxed{A = 7,\ \hat{\omega}_0 = 0.9\pi,\ \varphi = \pi/2}$

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(b) $f_s = 5$ samples/sec

$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{5}\right) = 7\cos\left(\frac{11\pi n}{5} - \pi/2\right)$

$= 7\cos\left(\frac{\pi n}{5} - \frac{\pi}{2}\right)$

$\boxed{A = 7,\ \hat{\omega}_0 = \frac{\pi}{5},\ \varphi = -\frac{\pi}{2}}$

(c) $f_s = 15$ samples/sec

$x(t)\Big|_{t=n/f_s} = x\left(\frac{n}{15}\right) = 7\cos\left(\frac{11\pi n}{15} - \frac{\pi}{2}\right)$

$A = 7,\ \hat{\omega}_0 = \frac{11\pi}{15} = 2\pi\left(\frac{5.5}{15}\right)$ & $\varphi = -\pi/2$

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P-4.3 Suppose that a discrete-time signal $x[n]$ is given by the formula

$$x[n] = 2.2\cos(0.3\pi n - \pi/3)$$

and that it was obtained by sampling a continuous-time signal $x(t) = A\cos(2\pi f_0 t + \phi)$ at a sampling rate of $f_s = 6000$ samples/sec. Determine three different continuous-time signals that could have produced $x[n]$. All these continuous-time signals should have a frequency less than 8 kHz. Write the mathematical formula for all three.

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$x[n] = 2.2\cos(0.3\pi n - \pi/3)$ $\boxed{f_s = 6000}$

Compare to

$x(\frac{n}{f_s}) = A\cos(2\pi f_0 \frac{n}{f_s} + \varphi)$ ← sampled continuous-time signal.

$\Rightarrow \frac{2\pi f_0}{f_s} = 0.3\pi$, or $0.3\pi + 2\pi$, or $0.3\pi - 2\pi$.

Solve: $\frac{2\pi f_0}{f_s} = 0.3\pi \Rightarrow f_0 = f_s\left(\frac{0.3}{2}\right) = 6000 \times 0.15$

$f_0 = 900$ Hz

$\rightarrow x(t) = 2.2\cos(1800\pi t - \pi/3)$

NOTE: difference is f_s

Then $\frac{2\pi f_0}{f_s} = 2.3\pi \Rightarrow f_0 = f_s\left(\frac{2.3}{2}\right) = 6900$ Hz

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$\rightarrow x(t) = 2.2\cos(2\pi(6900)t - \pi/3)$.

Finally,

$\frac{2\pi f_0}{f_s} = -1.7\pi \Rightarrow f_0 = f_s\left(\frac{-1.7}{2}\right) = -5100$ Hz.

$x(t) = 2.2\cos(2\pi(-5100)t - \pi/3)$

$\rightarrow x(t) = 2.2\cos(2\pi(5100)t + \pi/3)$

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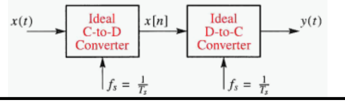
P-4.4 An amplitude-modulated (AM) cosine wave is represented by the formula

$$x(t) = [10 + \cos(2\pi(2000)t)]\cos(2\pi(10^4)t)$$

(a) Sketch the two-sided spectrum of this signal. Be sure to label important features of the plot.

(b) Is this waveform periodic? If so, what is the period?

(c) What relation must the sampling rate f_s satisfy so that $y(t) = x(t)$ in Fig. 4-26?



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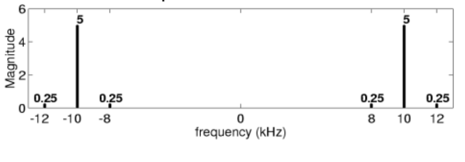
(a) $x(t) = \left[10 + \frac{1}{2}e^{j2\pi(2000)t} + \frac{1}{2}e^{-j2\pi(2000)t}\right]\left(\frac{1}{2}e^{j2\pi\times10^4 t} + \frac{1}{2}e^{-j2\pi\times10^4 t}\right)$

There are six terms:

$$x(t) = 5e^{j2\pi\times10^4 t} + 5e^{-j2\pi\times10^4 t} + \frac{1}{4}e^{j2\pi(12000)t} + \frac{1}{4}e^{-j2\pi(12000)t} + \frac{1}{4}e^{j2\pi(8000)t} + \frac{1}{4}e^{-j2\pi(8000)t}$$

Spectrum plot was created in MATLAB:



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(b) Yes the waveform is periodic. The six frequencies $\{-12,000, -10,000, -8000, 8000, 10000, 12000\}$ are all divisible by 2000 Hz. Therefore, $f_0 = 2000$ Hz is the fundamental frequency. The period is $1/f_0 = 1/2000$ sec $= \frac{1}{2}$ msec

(c) The sampling rate must be greater than twice the highest frequency in $x(t)$.

$\Rightarrow f_s > 2(12,000) = 24,000$ Hz

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P-4.5 Suppose that a discrete-time signal $x[n]$ is given by the formula

$$x[n] = 10\cos(0.2\pi n - \pi/7)$$

and that it was obtained by sampling a continuous-time signal at a sampling rate of $f_s = 1000$ samples/sec.

(a) Determine two *different* continuous-time signals $x_1(t)$ and $x_2(t)$ whose samples are equal to $x[n]$; i.e., find $x_1(t)$ and $x_2(t)$ such that $x[n] = x_1(nT_s) = x_2(nT_s)$ if $T_s = 0.001$. Both of these signals should have a frequency less than 1000 Hz. Give a formula for each signal.

(b) If $x[n]$ is given by the equation above, what signal will be reconstructed by an ideal D-to-C converter operating at sampling rate of 2000 samples/sec? That is, what is the output $y(t)$ in Fig. 4-26 if $x[n]$ is as given above?

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(a) Let $x(t) = 10\cos(\omega_0 t + \varphi)$

Sampling at a rate of $f_s \Rightarrow x[n] = x(t)\big|_{t=n/f_s} = x\left(\frac{n}{f_s}\right)$

$x[n] = 10\cos\left(\omega_0 \frac{n}{f_s} + \varphi\right)$ $\Rightarrow$ $\frac{\omega_0}{f_s} = 0.2\pi \Rightarrow \omega_0 = 0.2\pi \times 1000 = 200\pi$ ($f_s = 1000$)

Equate this to

$x[n] = 10\cos(0.2\pi n - \pi/7)$ $\qquad \varphi = -\pi/7$

A second possible signal is the "folded alias" at $(f_s - f_0)$

$f_s - f_0 = f_s - \frac{\omega_0}{2\pi} = 1000 - \frac{200\pi}{2\pi} = 900\,\text{Hz}$

In this case, the phase (φ) changes.

$\tilde{x}(t) = 10\cos(2\pi(f_s - f_0)t + \psi)$

$\tilde{x}[n] = 10\cos\left(2\pi(f_s - f_0)\frac{n}{f_s} + \psi\right) = 10\cos\left(2\pi n - 2\pi f_0 \frac{n}{f_s} + \psi\right)$

$= 10\cos\left(-2\pi \frac{f_0 n}{f_s} + \psi\right) = 10\cos\left(2\pi \frac{f_0}{f_s} n - \psi\right)$.

f_0 is still 100 Hz

$\Rightarrow \psi = +\pi/7$

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(b) Reconstruction of $x[n]$ with $f_s = 2000$ samples/sec.

The discrete and continuous domains are related by: $\frac{n}{f_s} \leftrightarrow t$ or $n \leftrightarrow f_s t$

So we replace "n" in $x[n]$ with $f_s t$. This is what an ideal D-to-A would do.

$x[n] = 10\cos(0.2\pi n - \pi/7)$

$x(t) = 10\cos(0.2\pi f_s t - \pi/7)$ $\leftarrow f_s = 2000$

$= 10\cos(400\pi t - \pi/7)$

$\omega_0 = 400\pi \Rightarrow f_0 = 200\,\text{Hz}$.

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P-4.8 The spectrum diagram gives the frequency content of a signal.

(a) Draw a sketch of the spectrum of

$$x(t) = \cos(50\pi t)\,\sin(700\pi t)$$

Label the frequencies and complex amplitudes of each component.

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without aliasing for any of the components.

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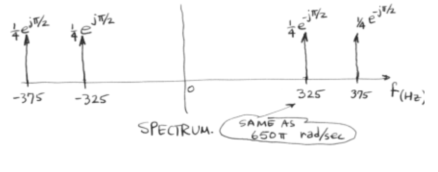
$x(t) = \cos(50\pi t)\sin(700\pi t)$

(a) $x(t) = \left(\frac{1}{2}e^{j50\pi t} + \frac{1}{2}e^{-j50\pi t}\right)\left(\frac{1}{2j}e^{j700\pi t} - \frac{1}{2j}e^{-j700\pi t}\right)$

$= \frac{1}{4j}e^{j750\pi t} + \frac{1}{4j}e^{j650\pi t} - \frac{1}{4j}e^{-j650\pi t} - \frac{1}{4j}e^{-j750\pi t}$

SAME AS $\frac{1}{4}e^{-j\pi/2}$ $\qquad$ SAME AS $\frac{1}{4}e^{+j\pi/2}$



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(b) Sampling Thm says sample at a rate greater than $\boxed{\text{two}}$ $\boxed{\text{times}}$ the highest freq.

HIGHEST FREQ = 375 Hz

$\Rightarrow f_s \geq 750\,\text{Hz}$.

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P-4.9 The spectrum diagram gives the frequency content of a signal.

(a) Draw a sketch of the spectrum of

$$x(t) = \sin^3(400\pi t)$$

Label the frequencies and complex amplitudes of each component.

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without aliasing for any of its components.

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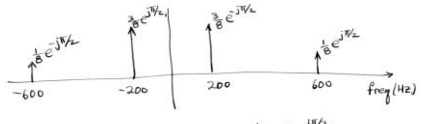
(a) Draw a sketch of the spectrum of $x(t)$ which is "sine-cubed" $x(t) = \sin^3(400\pi t)$

$$x(t) = \left(\frac{e^{j400\pi t} - e^{-j400\pi t}}{2j}\right)^3$$

$$= \frac{1}{-8j}\left\{ e^{j1200\pi t} - 3e^{j400\pi t} + 3e^{-j400\pi t} - e^{-j1200\pi t} \right).$$



$$\frac{1}{-8j} = \frac{1}{8}e^{j\pi/2}$$

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$f_s \geq 2 f_{HIGH}$$

$$\Rightarrow f_s \geq 1200\,\text{Hz}$$

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