Digital Signal Processing

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Digital Signal Processing

FIR Filtering

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LECTURE OBJECTIVES

- INTRODUCE FILTERING IDEA
  - Weighted Average
  - Running Average
- FINITE IMPULSE RESPONSE FILTERS
  - **FIR** Filters
  - Show how to **compute** the output $y[n]$ from the input signal, $x[n]$

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DIGITAL FILTERING

- CONCENTRATE on the COMPUTER
  - PROCESSING ALGORITHMS
  - SOFTWARE (MATLAB)
  - HARDWARE: DSP chips, VLSI
- DSP: DIGITAL SIGNAL PROCESSING

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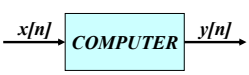
DISCRETE-TIME SYSTEM



- OPERATE on $x[n]$ to get $y[n]$
- WANT a **GENERAL** CLASS of SYSTEMS
  - **ANALYZE** the SYSTEM
    - TOOLS: TIME-DOMAIN & FREQUENCY-DOMAIN
  - **SYNTHESIZE** the SYSTEM

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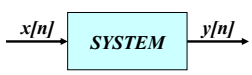
D-T SYSTEM EXAMPLES



- EXAMPLES:
  - POINTWISE OPERATORS
    - SQUARING: $y[n] = (x[n])^2$
  - RUNNING AVERAGE
    - RULE: "the output at time n is the average of three consecutive input values"

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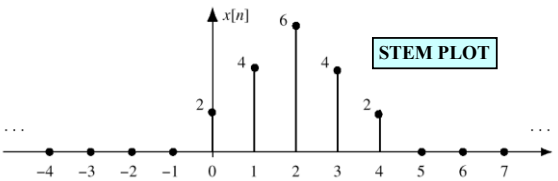
DISCRETE-TIME SIGNAL

- $x[n]$ is a LIST of NUMBERS
  - INDEXED by "n"



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3-PT AVERAGE SYSTEM

- ADD 3 CONSECUTIVE NUMBERS
  - Do this for each "n"

the following input–output equation

Make a TABLE

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0
$y[n]$	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

n=0 $y[0] = \frac{1}{3}(x[0] + x[1] + x[2])$

n=1 $y[1] = \frac{1}{3}(x[1] + x[2] + x[3])$

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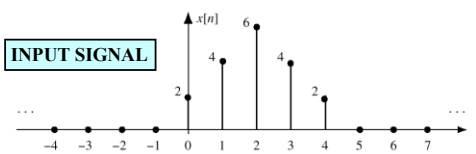

Figure 5.2 Finite-length input signal, $x[n]$.

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

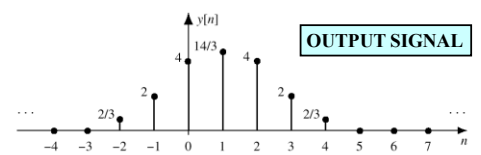

Figure 5.3 Output of running average, $y[n]$.

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PAST, PRESENT, FUTURE

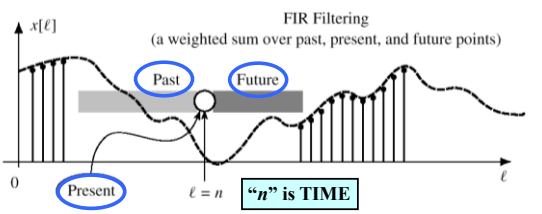

Figure 5.4 The running-average filter calculation at time index n uses values within a sliding window (shaded). Dark shading indicates the future ($\ell > n$); light shading, the past ($\ell < n$).

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ANOTHER 3-pt AVERAGER

- Uses "PAST" VALUES of $x[n]$
  - IMPORTANT IF "n" represents REAL TIME
    - WHEN $x[n]$ & $y[n]$ ARE STREAMS

$$y[n] = \tfrac{1}{3}(x[n] + x[n-1] + x[n-2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	6	7	$n > 7$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0	0	0
$y[n]$	0	0	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

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GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - For example, $b_k = \{3, -1, 2, 1\}$

$$y[n] = \sum_{k=0}^{3} b_k x[n-k]$$
$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

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GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- FILTER **ORDER** is M
- FILTER **LENGTH** is $L = M+1$
  - NUMBER of FILTER COEFFS is L

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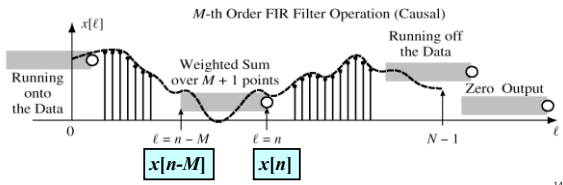
GENERAL FIR FILTER

- SLIDE a WINDOW across $x[n]$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



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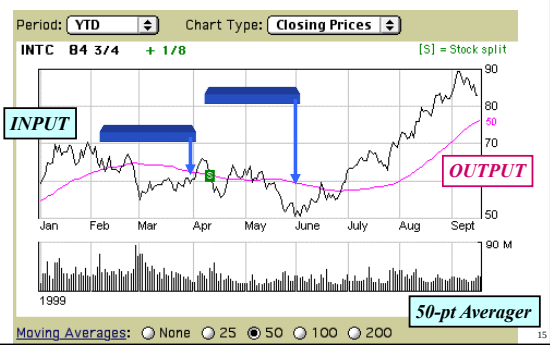
FILTERED STOCK SIGNAL



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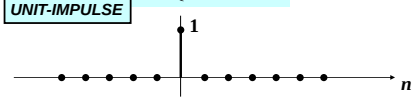
SPECIAL INPUT SIGNALS

- $x[n]$ = SINUSOID **FREQUENCY RESPONSE (LATER)**
- $x[n]$ has only one NON-ZERO VALUE

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

UNIT-IMPULSE



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UNIT IMPULSE SIGNAL $\delta[n]$

n	...	−2	−1	0	1	2	3	4	5	6	...
$\delta[n]$	0	0	0	1	0	0	0	0	0	0	0
$\delta[n-3]$	0	0	0	0	0	0	1	0	0	0	0

δ[n] is NON-ZERO When its argument is equal to ZERO

$n = 3$

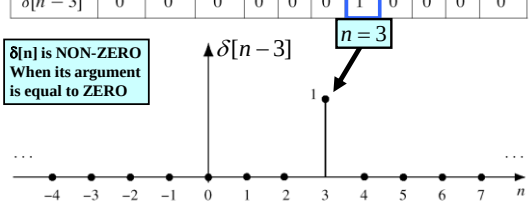

Figure 5.7 Shifted impulse sequence, $\delta[n-3]$.

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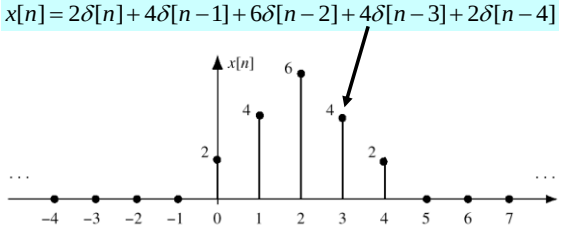
MATH FORMULA for $x[n]$

- Use SHIFTED IMPULSES to write $x[n]$

$$x[n] = 2\delta[n] + 4\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$



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SUM of SHIFTED IMPULSES

n	...	-2	-1	0	1	2	3	4	5	6	...
$2\delta[n]$	0	0	0	2	0	0	0	0	0	0	0
$4\delta[n-1]$	0	0	0	0	4	0	0	0	0	0	0
$6\delta[n-2]$	0	0	0	0	0	6	0	0	0	0	0
$4\delta[n-3]$	0	0	0	0	0	0	4	0	0	0	0
$2\delta[n-4]$	0	0	0	0	0	0	0	2	0	0	0
$x[n]$	0	0	0	2	4	6	4	2	0	0	0

$$x[n] = \sum_k x[k]\delta[n-k]$$

This formula ALWAYS works

$$= \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots \quad (5.3.6)$$

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4-pt AVERAGER

- CAUSAL SYSTEM: USE PAST VALUES

$$y[n] = \tfrac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$$

- INPUT = UNIT IMPULSE SIGNAL = $\boldsymbol{\delta[n]}$

$$x[n] = \delta[n]$$
$$y[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{4}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3]$$

- OUTPUT is called "IMPULSE RESPONSE"

$$h[n] = \{\ldots, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, \ldots\}$$

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4-pt Avg Impulse Response

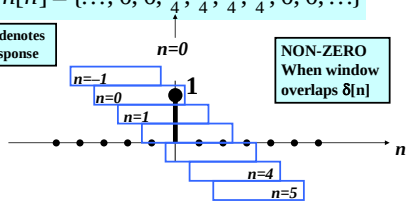
$$y[n] = \tfrac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$$

$\delta[n]$ "READS OUT" the FILTER COEFFICIENTS

$$h[n] = \{\ldots, 0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, \ldots\}$$

"h" in h[n] denotes Impulse Response

NON-ZERO When window overlaps δ[n]



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FIR IMPULSE RESPONSE

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

n	$n<0$	0	1	2	3	...	M	$M+1$	$n>M+1$
$x[n]=\delta[n]$	0	1	0	0	0	0	0	0	0
$y[n]=h[n]$	0	b_0	b_1	b_2	b_3	...	b_M	0	0

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

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FILTERING EXAMPLE

- 7-point AVERAGER
  - Removes cosine
    - By making its amplitude (A) smaller

$$y_7[n] = \sum_{k=0}^{6} \left(\tfrac{1}{7}\right) x[n-k]$$

- 3-point AVERAGER
  - Changes A slightly

$$y_3[n] = \sum_{k=0}^{2} \left(\tfrac{1}{3}\right) x[n-k]$$

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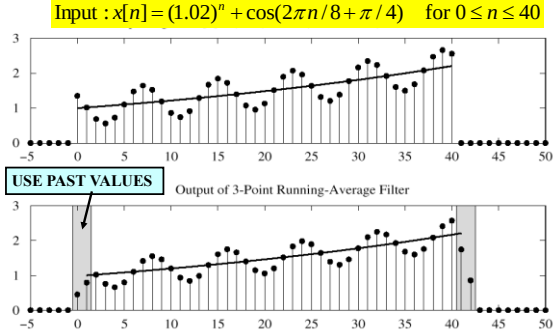
3-pt AVG EXAMPLE

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4)$ for $0 \le n \le 40$



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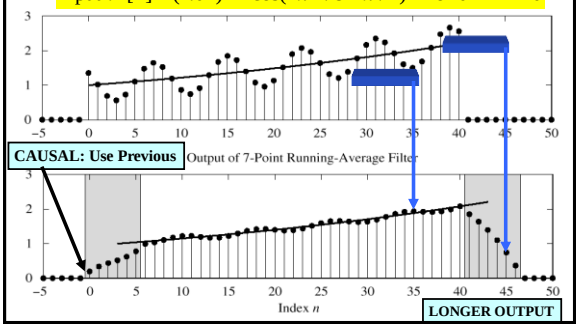
7-pt FIR EXAMPLE (AVG)

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4)$ for $0 \le n \le 40$



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Digital Signal Processing

Linearity & Time-Invariance

Convolution

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LECTURE OBJECTIVES

- GENERAL PROPERTIES of FILTERS
  - LINEARITY
  - TIME-INVARIANCE
  - ==> **CONVOLUTION**

  LTI SYSTEMS
  - BLOCK DIAGRAM REPRESENTATION
    - Components for Hardware
    - Connect Simple Filters Together to Build More Complicated Systems

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OVERVIEW

- IMPULSE RESPONSE, $h[n]$
  - FIR case: same as $\{b_k\}$
- CONVOLUTION
  - GENERAL: $y[n] = h[n] * x[n]$
  - GENERAL CLASS of SYSTEMS
  - LINEAR and TIME-INVARIANT
- ALL **LTI** systems have $h[n]$ & use convolution

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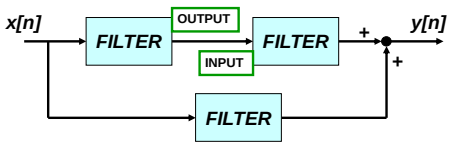
DIGITAL FILTERING

- CONCENTRATE on the FILTER (DSP)
- DISCRETE-TIME SIGNALS
  - FUNCTIONS of n, the "time index"
  - INPUT $x[n]$
  - OUTPUT $y[n]$

BUILDING BLOCKS



- BUILD UP COMPLICATED FILTERS
  - FROM SIMPLE MODULES
  - Ex: FILTER MODULE MIGHT BE 3-pt FIR

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GENERAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - For example, $b_k = \{3, -1, 2, 1\}$

$$y[n] = \sum_{k=0}^{3} b_k x[n-k]$$
$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

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MATLAB for FIR FILTER

- `yy = conv(bb,xx)`
  - VECTOR **bb** contains Filter Coefficients
  - DSP-First: `yy = firfilt(bb,xx)`
- FILTER COEFFICIENTS $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

`conv2()` *for images*

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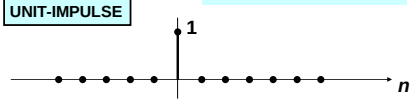
SPECIAL INPUT SIGNALS

- $x[n]$ = SINUSOID **FREQUENCY RESPONSE**
- $x[n]$ has only one NON-ZERO VALUE

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

UNIT-IMPULSE



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FIR IMPULSE RESPONSE

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

n	$n<0$	0	1	2	3	...	M	$M+1$	$n>M+1$
$x[n]=\delta[n]$	0	1	0	0	0	0	0	0	0
$y[n]=h[n]$	0	b_0	b_1	b_2	b_3	...	b_M	0	0

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

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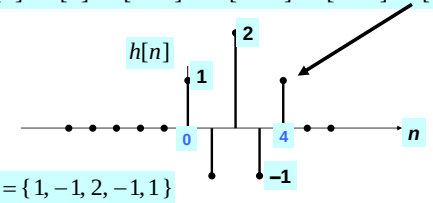
MATH FORMULA for *h*[*n*]

- Use SHIFTED IMPULSES to write ***h***[***n***]

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$



$b_k = \{1, -1, 2, -1, 1\}$

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LTI: Convolution Sum

- **Output = Convolution of $x[n]$ & $h[n]$**
  - NOTATION: $y[n] = h[n] * x[n]$
  - Here is the FIR case:

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Same as b_k

FINITE LIMITS

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CONVOLUTION Example

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$
$$x[n] = u[n]$$

n	−1	0	1	2	3	4	5	6	7
$x[n]$	0	1	1	1	1	1	1	1	...
$h[n]$	0	1	−1	2	−1	1	0	0	0
$h[0]x[n]$	0	1	1	1	1	1	1	1	1
$h[1]x[n-1]$	0	0	−1	−1	−1	−1	−1	−1	−1
$h[2]x[n-2]$	0	0	0	2	2	2	2	2	2
$h[3]x[n-3]$	0	0	0	0	−1	−1	−1	−1	−1
$h[4]x[n-4]$	0	0	0	0	0	1	1	1	1
$y[n]$	0	1	0	2	1	2	2	2	...

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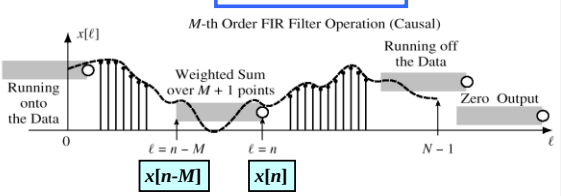
GENERAL FIR FILTER

- SLIDE a Length-L WINDOW over x[n]

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



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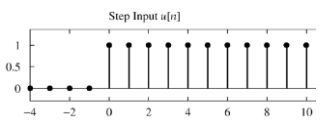
POP QUIZ

- FIR Filter is "FIRST DIFFERENCE"
  - **$y[n] = x[n] - x[n-1]$**
- INPUT is "UNIT STEP"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- Find $y[n]$

$$y[n] = u[n] - u[n-1] = \delta[n]$$

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HARDWARE STRUCTURES

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- INTERNAL STRUCTURE of "FILTER"
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE "HOOK" THEM TOGETHER?
- SIGNAL FLOW GRAPH NOTATION

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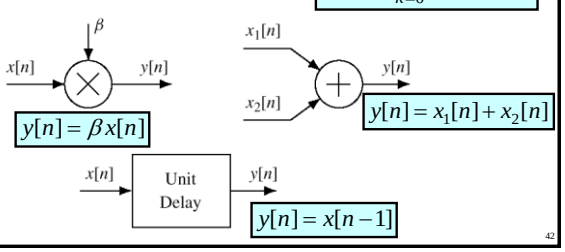
HARDWARE ATOMS

- Add, Multiply & Store

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



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FIR STRUCTURE

- Direct Form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

SIGNAL FLOW GRAPH

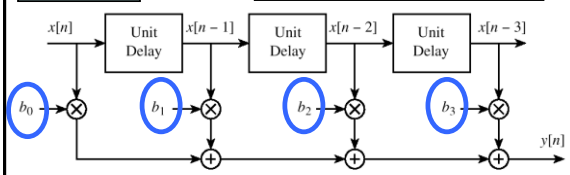

Figure 5.13 Block-diagram structure for the Mth order FIR filter.

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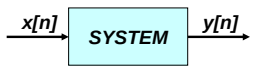
SYSTEM PROPERTIES



- MATHEMATICAL DESCRIPTION
- TIME-INVARIANCE
- LINEARITY
- CAUSALITY
  - "No output prior to input"

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TIME-INVARIANCE

- IDEA:
  - "Time-Shifting the input will cause the same time-shift in the output"
- EQUIVALENTLY,
  - We can prove that
    - The time origin (n=0) is picked arbitrary

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TESTING Time-Invariance

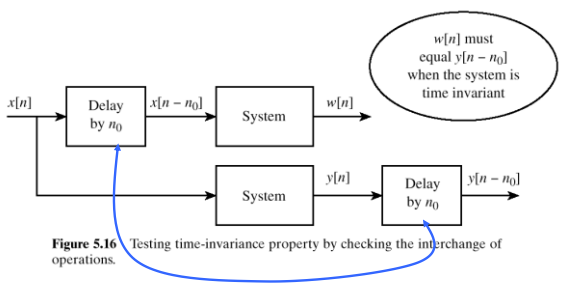

Figure 5.16 Testing time-invariance property by checking the interchange of operations.

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LINEAR SYSTEM

- LINEARITY = Two Properties
- SCALING
  - "Doubling $x[n]$ will double $y[n]$"
- SUPERPOSITION:
  - "Adding two inputs gives an output that is the sum of the individual outputs"

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TESTING LINEARITY

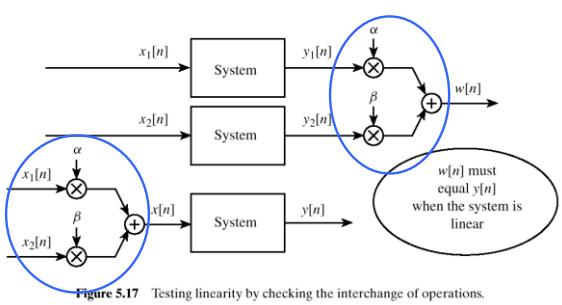

Figure 5.17 Testing linearity by checking the interchange of operations.

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LTI SYSTEMS

- LTI: Linear & Time-Invariant
- COMPLETELY CHARACTERIZED by:
  - **IMPULSE RESPONSE** $h[n]$
  - **CONVOLUTION**: $y[n] = x[n]*h[n]$
    - The "rule" defining the system can ALWAYS be re-written as convolution
- FIR Example: $h[n]$ is same as b_k

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POP QUIZ

- FIR Filter is "FIRST DIFFERENCE"
  - $y[n] = x[n] - x[n-1]$
- Write output as a convolution
  - Need impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

  - Then, another way to compute the output:

$$y[n] = (\delta[n] - \delta[n-1]) * x[n]$$

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CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$

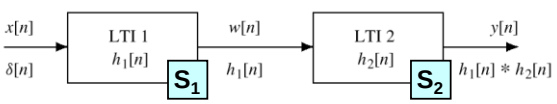

Figure 5.19 A Cascade of Two LTI Systems.

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CASCADE EQUIVALENT

- Find "overall" $h[n]$ for a cascade ?

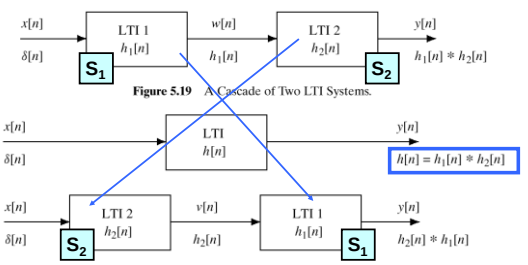

Figure 5.19 A Cascade of Two LTI Systems.

Figure 5.20 Switching the order of cascaded LTI systems.

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Digital Signal Processing

Frequency Response of FIR Filters

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LECTURE OBJECTIVES

- SINUSOIDAL INPUT SIGNAL
  - DETERMINE the FIR FILTER OUTPUT
- FREQUENCY RESPONSE of FIR
  - PLOTTING vs. Frequency
  - MAGNITUDE vs. Freq
  - PHASE vs. Freq

MAG

PHASE

$$H(e^{j\hat{\omega}}) = \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

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DOMAINS: Time & Frequency

- **Time-Domain: "*n*" = time**
  - $x[n]$ discrete-time signal
  - $x(t)$ continuous-time signal
- **Frequency Domain (sum of sinusoids)**
  - Spectrum vs. f (Hz)
    - ANALOG vs. DIGITAL
  - Spectrum vs. omega-hat
- Move back and forth QUICKLY

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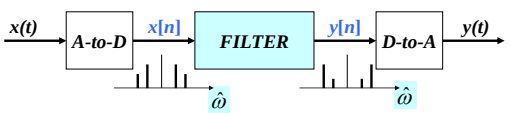
DIGITAL "FILTERING"



- CONCENTRATE on the **SPECTRUM**
- SINUSOIDAL INPUT
  - INPUT $x[n]$ = SUM of SINUSOIDS
  - Then, OUTPUT $y[n]$ = SUM of SINUSOIDS

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SINUSOIDAL RESPONSE

- INPUT: x[n] = SINUSOID
- OUTPUT: y[n] will also be a SINUSOID
  - Different Amplitude and Phase
  - **SAME** Frequency
- AMPLITUDE & PHASE CHANGE
  - Called the **FREQUENCY RESPONSE**

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COMPLEX EXPONENTIAL

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

x[n] is the input signal—a complex exponential

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR DIFFERENCE EQUATION

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COMPLEX EXP OUTPUT

- Use the FIR "Difference Equation"

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k Ae^{j\varphi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{M} b_k e^{j\hat{\omega}(-k)}\right) Ae^{j\varphi}e^{j\hat{\omega}n}$$

$$= H(\hat{\omega})Ae^{j\varphi}e^{j\hat{\omega}n}$$

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FREQUENCY RESPONSE

- At each frequency, we can DEFINE

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

FREQUENCY RESPONSE

- Complex-valued formula
  - Has MAGNITUDE vs. frequency
  - And PHASE vs. frequency
- Notation: $H(e^{j\hat{\omega}})$ in place of $H(\hat{\omega})$

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EXAMPLE 6.1

$\{b_k\} = \{1,\ 2,\ 1\}$

$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$
$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}})$$
$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

EXPLOIT SYMMETRY

Since $(2 + 2\cos\hat{\omega}) \geq 0$

Magnitude is $|H(e^{j\hat{\omega}})| = (2 + 2\cos\hat{\omega})$

and Phase is $\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$

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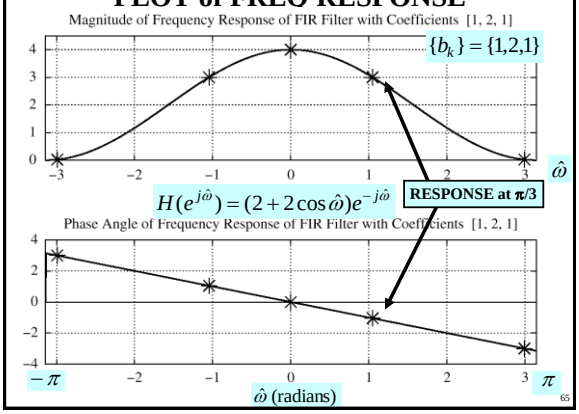
PLOT of FREQ RESPONSE



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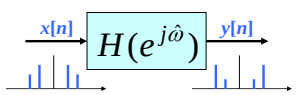
EXAMPLE 6.2

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known

and $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$



$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

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EXAMPLE 6.2 (answer)

Find $y[n]$ when $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$

One Step - evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \pi/3$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$H(e^{j\hat{\omega}}) = 3e^{-j\pi/3} \quad @\ \hat{\omega} = \pi/3$$

$$y[n] = \left(3e^{-j\pi/3}\right) \times 2e^{j\pi/4}e^{j(\pi/3)n} = 6e^{-j\pi/12}e^{j(\pi/3)n}$$

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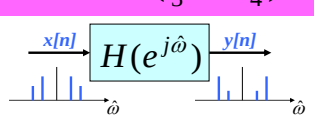
EXAMPLE: COSINE INPUT

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known

and $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$



$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

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EX: COSINE INPUT

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$2\cos(\tfrac{\pi}{3}n + \tfrac{\pi}{4}) = e^{j(\pi n/3+\pi/4)} + e^{-j(\pi n/3+\pi/4)}$$
$$\Rightarrow x[n] = x_1[n] + x_2[n]$$

Use Linearity

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3+\pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3+\pi/4)}$$
$$\Rightarrow y[n] = y_1[n] + y_2[n]$$

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EX: COSINE INPUT (ans-2)

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3+\pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3+\pi/4)}$$
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3+\pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3+\pi/4)}$$

$$y[n] = 3e^{j(\pi n/3-\pi/12)} + 3e^{-j(\pi n/3-\pi/12)}$$
$$\Rightarrow y[n] = 6\cos(\tfrac{\pi}{3}n - \tfrac{\pi}{12})$$

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MATLAB: FREQUENCY RESPONSE

- `HH = freqz(bb,1,ww)`
  - VECTOR `bb` contains Filter Coefficients
  - SP-First: `HH = freekz(bb,1,ww)`
- FILTER COEFFICIENTS $\{b_k\}$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

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Time & Frequency Relation

- Get Frequency Response from $h[n]$
  - Here is the FIR case:

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

IMPULSE RESPONSE

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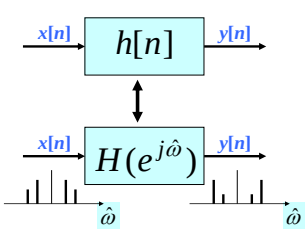
BLOCK DIAGRAMS

- Equivalent Representations



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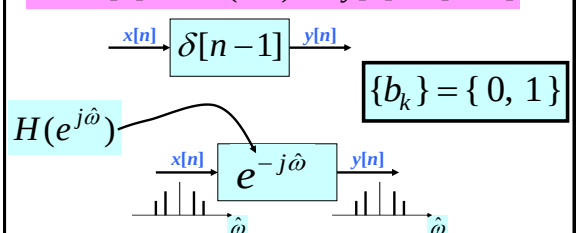
UNIT-DELAY SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-1]$



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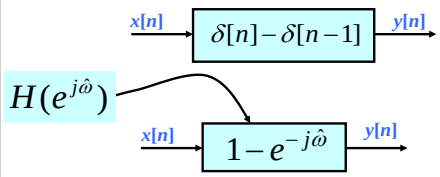
FIRST DIFFERENCE SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for the Difference Equation : $y[n] = x[n] - x[n-1]$



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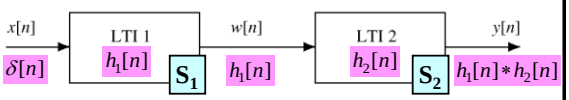
CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$
  - WHAT is the overall FREQUENCY RESPONSE ?



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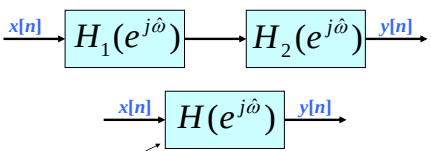
CASCADE EQUIVALENT

- MULTIPLY the Frequency Responses



EQUIVALENT SYSTEM

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$

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Digital Signal Processing

Z Transforms: Introduction

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LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
  - Give Mathematical Definition
  - Show how the H(z) POLYNOMIAL simplifies analysis
    - CONVOLUTION is SIMPLIFIED !
- Z-Transform can be applied to
  - FIR Filter: $h[n] \rightarrow H(z)$
  - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_n h[n]z^{-n}$$

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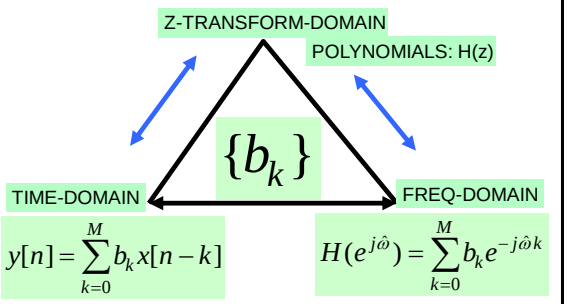
THREE DOMAINS



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Three main reasons for Z-Transform

- Offers compact and convenient notation for describing digital signals and systems
- Widely used by DSP designers, and in the DSP literature
- Pole-zero description of a processor is a great help in visualizing its stability and frequency response characteristic

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TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are EASIER & FAMILIAR
  - Use POLYNOMIALS
- TRANSFORM both ways
  - $x[n] \rightarrow X(z)$ (into the z domain)
  - $X(z) \rightarrow x[n]$ (back to the time domain)

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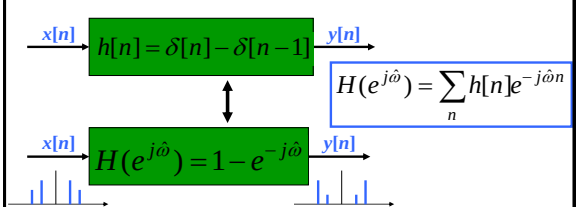
"TRANSFORM" EXAMPLE

- Equivalent Representations



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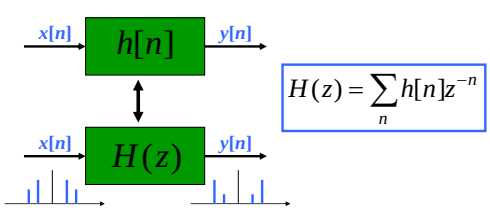
Z-TRANSFORM IDEA 2.12.13

- **POLYNOMIAL** REPRESENTATION



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Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $H(z)=\sum_n h[n]z^{-n}$

APPLIES to Any SIGNAL

- EXAMPLE:

$\{h[n]\}=\{2,0,-3,0,2\}$

$$H(z)=2z^{-0}+0z^{-1}-3z^{-2}+0z^{-3}+2z^{-4}$$
$$=2-3z^{-2}+2z^{-4}$$
$$=2-3(z^{-1})^2+2(z^{-1})^4$$

POLYNOMIAL in z^{-1}

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Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n] z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$X(z) = ?$

$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$

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Example 7.2

$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$x[n] = ?$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

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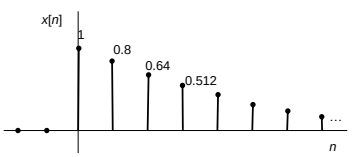
Example

- Find the Z-Transform of the exponentially decaying signal shown in the following figure, expressing it as compact as possible.



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- The geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

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- The Z-Transform of the signal:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= 1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \cdots$$

$$= 1 + 0.8(z^{-1}) + 0.64(z^{-1})^2 + 0.512(z^{-1})^3 + \cdots$$

$$= 1 + (0.8z^{-1}) + (0.8z^{-1})^2 + (0.8z^{-1})^3 + \cdots$$

$$= \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

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Example

- Find and sketch, the signal corresponding to the Z-Transform:

$$X(z) = \frac{1}{z + 1.2}$$

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- Recasting X(z) as a power series in z^{-1}, we obtain:

$$X(z) = \frac{1}{(z+1.2)} = \frac{z^{-1}}{(1+1.2z^{-1})} = z^{-1}(1+1.2z^{-1})^{-1}$$
$$= z^{-1}\{1+(-1.2z^{-1})+(-1.2z^{-1})^2+(-1.2z^{-1})^3+\cdots\}$$
$$= z^{-1} - 1.2z^{-2} + 1.44z^{-3} - 1.728z^{-4} + \cdots$$

- Succesive values of $x[n]$, starting at n=0, are therefore:

  0, 1, -1.2, 1.44, -1.728, $\cdots$

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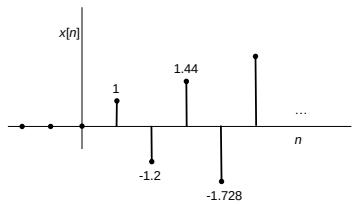
- $x[n]$ is shown in the following figure:



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Z-Transform of FIR Filter

- CALLED the **SYSTEM FUNCTION**
  - $h[n]$ is same as $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

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Z-Transform of FIR Filter

- Get $H(z)$ DIRECTLY from the $\{b_k\}$
- Example 7.3 in the book:

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

$$H(z) = \sum b_k z^{-1} = 6 - 5z^{-1} + z^{-2}$$

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Ex. DELAY SYSTEM

- UNIT DELAY: find $h[n]$ and $H(z)$

$$x[n] \rightarrow \boxed{\delta[n-1]} \rightarrow y[n] = x[n-1]$$

$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$

$$x[n] \rightarrow \boxed{z^{-1}} \rightarrow y[n]$$

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DELAY EXAMPLE

- UNIT DELAY: find $y[n]$ via polynomials
  - x[n] = {3,1,4,1,5,9,0,0,0,...}

$$Y(z) = z^{-1}X(z)$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

n	$n<0$	0	1	2	3	4	5	6	$n>6$
$y[n]$	0	0	3	1	4	1	5	9	0

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DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \iff z^{-1}X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \iff z^{-n_0}X(z)$$

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GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$ (for FIR, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$\text{and} \quad h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and} \quad H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

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FIR Filter = CONVOLUTION

```
x[n], X(z)     0   +1   -1   +1   -1
h[n], H(z)     1    2    3    4
               ----------------------------------------
               0   +1   -1   +1   -1
                    0   +2   -2   +2   -2
                         0   +3   -3   +3   -3
                              0   +4   -4   +4   -4
               ----------------------------------------
y[n], Y(z)     0   +1   +1   +2   +2   -3   +1   -4
```

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

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CONVOLUTION PROPERTY

- PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right)$$

MULTIPLY Z-TRANSFORMS

$$= \left(\sum_{k=0}^{M} h[k]z^{-k}\right) X(z) = H(z)X(z).$$

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CONVOLUTION EXAMPLE

- **MULTIPLY** the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$\text{and} \quad h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and} \quad H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

MULTIPLY $H(z)X(z)$

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CONVOLUTION EXAMPLE

- Finite-Length input $x[n]$
- FIR Filter (L=4)

MULTIPLY Z-TRANSFORMS

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1+2)z^{-2} + (1-2+3)z^{-3} + (-1+2-3+4)z^{-4}$$

$$+ \ (-2+3-4)z^{-5} + (-3+4)z^{-6} + (-4)z^{-7}$$

$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n]$ = ?

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CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - Remember: $h_1[n] * h_2[n]$
  - How to combine $H_1(z)$ and $H_2(z)$?

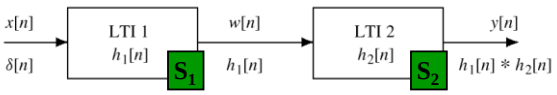

Figure 5.19 A Cascade of Two LTI Systems.

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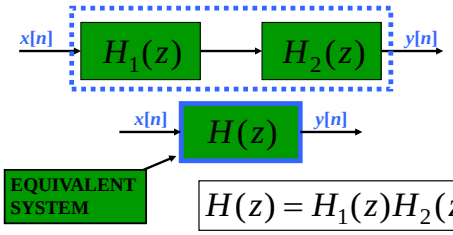
CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

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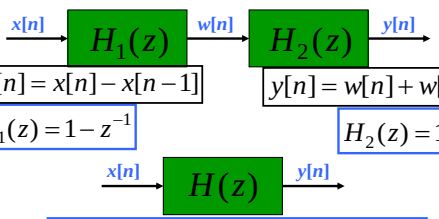
CASCADE EXAMPLE



$$w[n] = x[n] - x[n-1]$$

$$H_1(z) = 1 - z^{-1}$$

$$y[n] = w[n] + w[n-1]$$

$$H_2(z) = 1 + z^{-1}$$

$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n-2]$$

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Digital Signal Processing

IIR Filters:

Feedback and H(z)

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LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
  - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

– Show how to compute the output $y[n]$
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response $h[n] \leftrightarrow H(z)$

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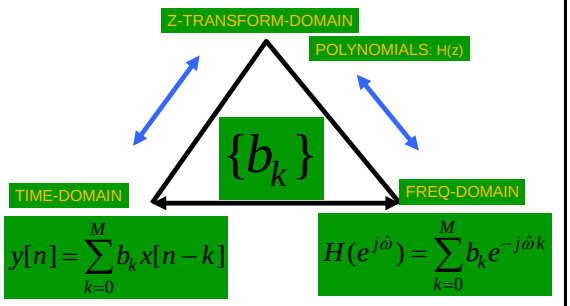
THREE DOMAINS



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Quick Review: Delay by n_d

$$y[n]=x[n-n_d]$$

IMPULSE RESPONSE $h[n]=\delta[n-n_d]$

SYSTEM FUNCTION $H(z)=z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}})=e^{-j\hat{\omega}n_d}$

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LOGICAL THREAD

- FIND the IMPULSE RESPONSE, $h[n]$
  - INFINITELY LONG
  - **IIR** Filters

$$H(z)=\sum_{n=0}^{\infty} h[n]z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n]=a_1 y[n-1]+b_0 x[n]+b_1 x[n-1]$$

PREVIOUS FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

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FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n]=0.8y[n-1]+3x[n]-2x[n-1]$$

FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

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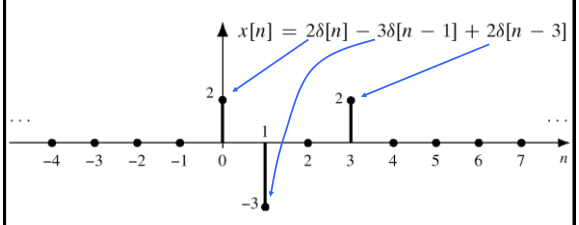
COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$



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COMPUTE $y[n]$

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED y[-1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

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AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

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COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

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COMPUTE MORE $y[n]$

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$
$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$
$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

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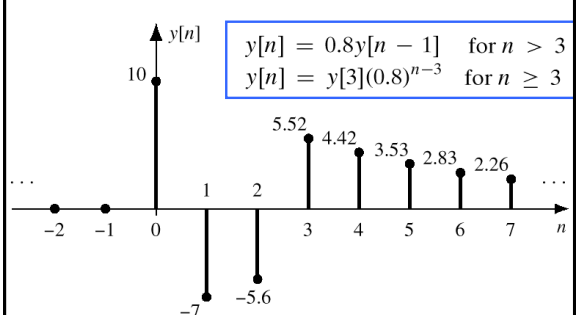
PLOT $y[n]$

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



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IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

$$h[n] = b_0(a_1)^n u[n]$$

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IMPULSE RESPONSE

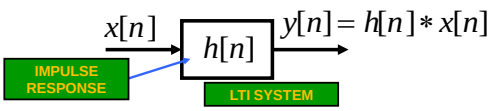
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

$$h[n] = 3(0.8)^n u[n]$$

- CONVOLUTION in TIME-DOMAIN



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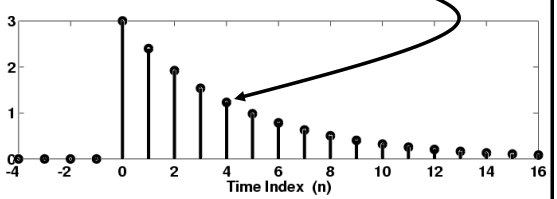
PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



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Infinite-Length Signal: *h*[*n*]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

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Derivation of *H*(*z*)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

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H(*z*) = z-Transform{ *h*[*n*] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

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H(*z*) = z-Transform{ *h*[*n*] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

z^{-1} is a shift

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

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CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

- CONVOLUTION in TIME-DOMAIN

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

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STEP RESPONSE: $x[n]=u[n]$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n]=1, \quad \text{for } n \geq 0$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

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DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

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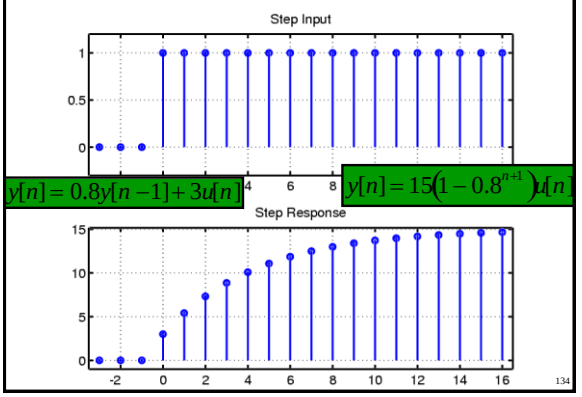
PLOT STEP RESPONSE



$y[n] = 0.8y[n-1] + 3u[n]$

$y[n] = 15(1 - 0.8^{n+1})u[n]$

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Digital Signal Processing

IIR Filters:

H(z) and Frequency Response

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LECTURE OBJECTIVES

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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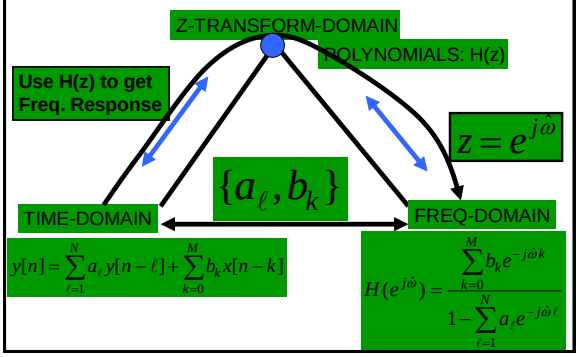
THREE DOMAINS



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H(*z*) = z-Transform{ *h*[*n*] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

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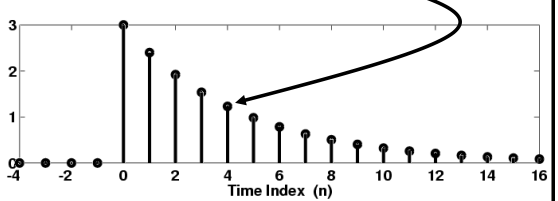
Typical IMPULSE Response

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



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First-Order Transform Pair

$$h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

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DELAY PROPERTY of *X*(*z*)

- DELAY in TIME<-->Multiply $X(z)$ by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

Proof: $\sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

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Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

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SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

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SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

- **READ** the FILTER COEFFS:

H(z)

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

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CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

- **CONVOLUTION** in TIME-DOMAIN

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

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POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) → inf

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EXAMPLE: Poles & Zeros

- VALUE of $H(z)$ at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

POLE at z=0.8

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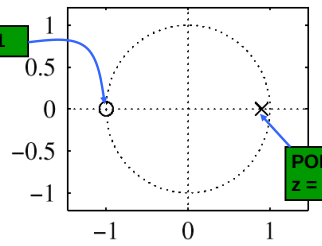
POLE-ZERO PLOT

ZERO at z = -1



POLE at z = 0.8

$$\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

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FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

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FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400, \quad @\,\hat{\omega}=\pi\,?$$

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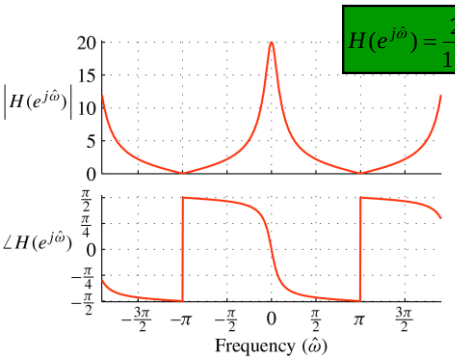
Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$



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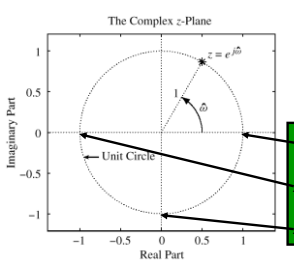
UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

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SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

$$\text{if} \quad x[n] = e^{j\hat{\omega} n}$$
$$\text{then} \quad y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}$$
$$\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

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POP QUIZ

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
  - When

$$x[n] = \cos(0.25\pi n)$$

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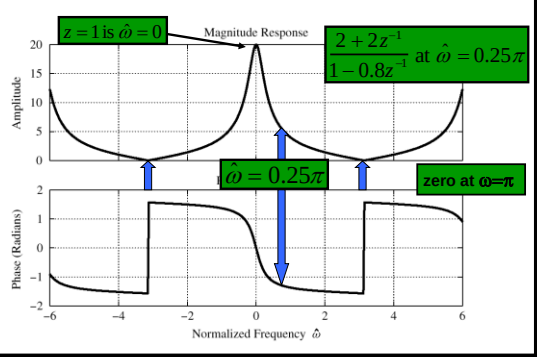
Evaluate FREQ. RESPONSE



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POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find output, $y[n]$, when
  - Evaluate at

$$x[n] = \cos(0.25\pi n)$$

$$z = e^{j0.25\pi}$$

$$H(z) = \frac{2+2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$

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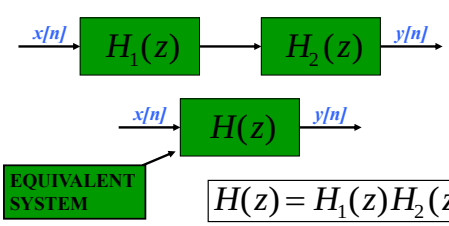
CASCADE EQUIVALENT

- Multiply the System Functions



$$H(z) = H_1(z)H_2(z)$$

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Digital Signal Processing

3-Domains for IIR

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LECTURE OBJECTIVES

- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- $H(z)$ can have COMPLEX POLES & ZEROS
- THREE-DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

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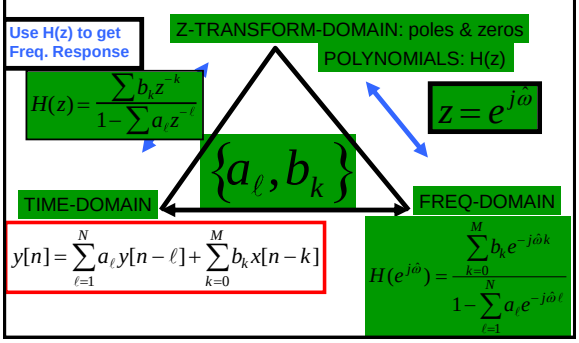
THREE DOMAINS



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Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\frac{1}{1 - az^{-1}}$

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SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

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MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

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TWO COMPLEX POLES

30.04.13

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

$$(p_k)^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make **BANDPASS FILTER**

$$\text{pole} = re^{j\theta}$$
$$r \to 1?$$

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2nd ORDER EXAMPLE

$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3}n)u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3})u[n]$$

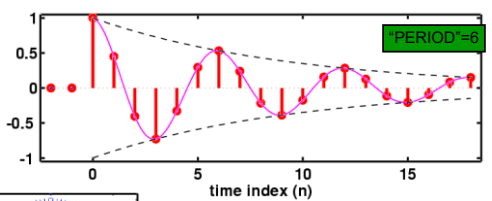
$$H(z) = \frac{0.5}{1-0.9e^{j\pi/3}z^{-1}} + \frac{0.5}{1-0.9e^{-j\pi/3}z^{-1}}$$

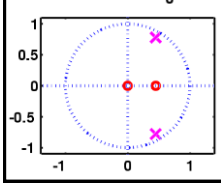
$$H(z) = \frac{1-0.9\cos(\tfrac{\pi}{3})z^{-1}}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})}$$

$$H(z) = \frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

191

$h[n]$: Decays & Oscillates





$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3}n)u[n]$$

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

192

2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n] = r^n \cos(\theta n)u[n]$$

$$H(z) = \frac{1-r\cos\theta z^{-1}}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

$$h[n] = Ar^n \cos(\theta n+\varphi)u[n]$$

$$H(z) = A\frac{\cos\varphi - r\cos(\theta-\varphi)z^{-1}}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

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2nd ORDER EX: n-Domain

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - 0.45x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```

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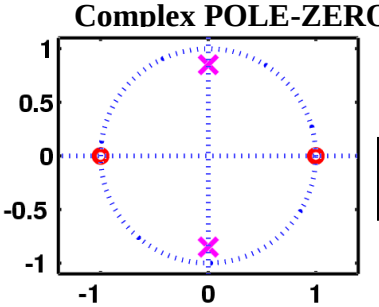
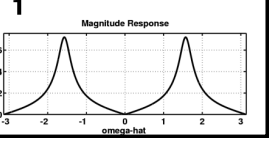
Complex POLE-ZERO PLOT



$$\frac{1-z^{-2}}{1+0.7225z^{-2}}$$



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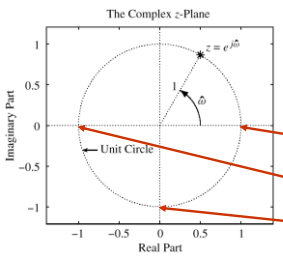
UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$
$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$
$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

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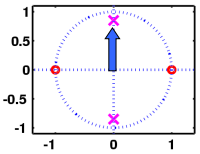
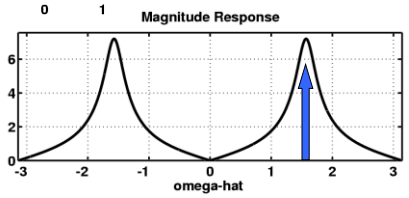
FREQUENCY RESPONSE from POLE-ZERO PLOT

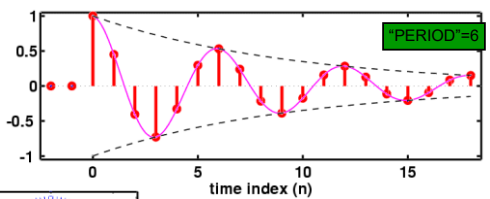


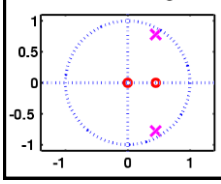
$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225e^{-j2\hat{\omega}}}$$



197

$h[n]$: Decays & Oscillates



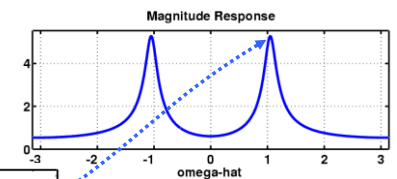


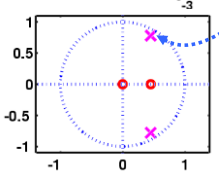
$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

198

Complex POLE-ZERO PLOT

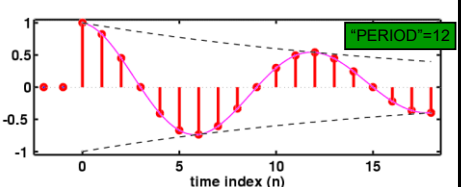


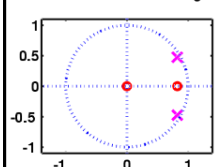


$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

199

h[n]: Decays & Oscillates





$$h[n] = (0.95)^n \cos(\tfrac{\pi}{6} n) u[n]$$

$$\frac{1 - 0.8227z^{-1}}{1 - 1.6454z^{-1} + 0.9025z^{-2}}$$

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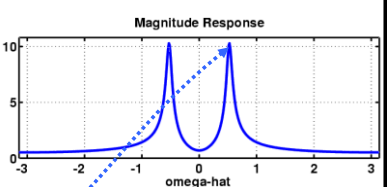
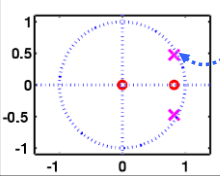
Complex POLE-ZERO PLOT





$$\frac{1 - 0.8227z^{-1}}{1 - 1.6454z^{-1} + 0.9025z^{-2}}$$

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THREE INPUTS

- Given: $H(z) = \dfrac{5}{1+0.8z^{-1}}$
- Find the output, $y[n]$
  - When

$$x[n] = \cos(0.2\pi n)$$
$$x[n] = u[n]$$
$$x[n] = \cos(0.2\pi n)u[n]$$

202

SINUSOID ANSWER

- Given: $H(z) = \dfrac{5}{1+0.8z^{-1}}$
- The input: $x[n] = \cos(0.2\pi n)$
- Then y[n] $y[n] = M\cos(0.2\pi n + \psi)$

$$H(e^{j0.2\pi}) = \frac{5}{1+0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$

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Step Response

$$Y(z) = H(z)X(z) = \left(\frac{5}{1+.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

Partial Fraction Expansion

$$Y(z) = \frac{A}{1+.8z^{-1}} + \frac{B}{1-z^{-1}} = \frac{(A+B)+(.8B-A)z^{-1}}{\left(1+.8z^{-1}\right)\left(1-z^{-1}\right)}$$

$$\Rightarrow \quad (A+B) = 5 \quad \text{and} \quad (.8B-A) = 0$$

$$Y(z) = \frac{A}{1+.8z^{-1}} + \frac{B}{1-z^{-1}}$$

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Step Response

$$Y(z) = \frac{\frac{20}{9}}{1+.8z^{-1}} + \frac{\frac{25}{9}}{1-z^{-1}}$$

$$y[n] = \frac{20}{9}(-.8)^n u[n] + \frac{25}{9}u[n]$$

$$y[n] \to \frac{25}{9} \quad \text{as} \quad n \to \infty$$

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Stability

- Nec. & suff. condition:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1-az^{-1}}$$

$$\sum_{n=0}^{\infty} |b||a|^n < \infty \text{ if } |a| < 1 \Rightarrow$$

Pole must be Inside unit circle

209

SINUSOID starting at *n*=0

- We'll look at an example in MATLAB
  - $\cos(0.2\pi n)$
  - Pole at –0.8, so a^n is $(-0.8)^n$
- There are two components:
  - TRANSIENT
    - Start-up region just after n=0; $(-0.8)^n$
  - STEADY-STATE
    - Eventually, $y[n]$ looks sinusoidal.
    - **Magnitude & Phase from Frequency Response**

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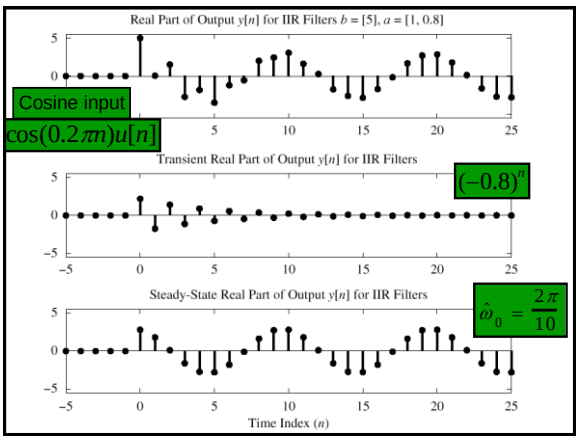

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STABILITY

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$

212

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

POLE @ z=1.1

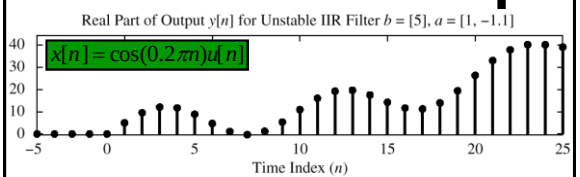

Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

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BONUS QUESTION

- Given: $H(z) = \dfrac{5}{1 + 0.8z^{-1}}$
- The input is $x[n] = 4\cos(\pi n - 0.5\pi)$
- Then find $y[n]$ $y[n] = ?$

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