BLM1612 - Circuit Theory

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2nd Order Circuits

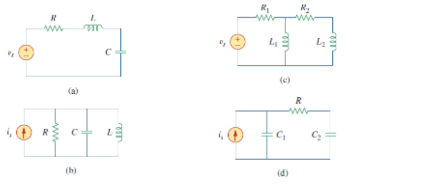
2^{nd} Order Circuits

Objective of Lecture

- Demonstrate how to determine the boundary conditions on the voltages and currents in a 2^{nd} order circuit.
  - These boundary conditions will be used when calculating the transient response of the circuit.

2^{nd} Order Circuits

- A second-order circuit is characterized by a second-order differential equation.
  - The circuit will contain at least one resistor and the equivalent of two energy storage elements
    - 2 capacitors, 2 inductors, or a capacitor and an inductor



Boundary Conditions

- Steady state
  - For step response functions $u(t-t_o)$ for all times between

    $t = +/- \infty$ except for some time period after $t = t_o$
    - Capacitors are open circuits
    - Inductors are short circuits
- During the transition at the step $t = t_o$
  - Voltage across a capacitor is continuous
    - $v_C(t_o^+) = v_C(t_o^-)$
  - Current through an inductor is continuous
    - $i_L(t_o^+) = i_L(t_o^-)$

Initial Condition

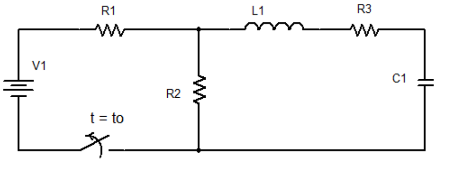
- Redraw the circuit at $t < t_o$
- Determine the value of all voltage and current sources at $t < t_o$
- Make the appropriate substitutions for the energy storage devices.
  - Substitute an open circuit ($\infty\Omega$ resistor) for all capacitors.
    - Note: $I_C(t < t_o) = 0A$.
  - Substitute an short circuit (0Ω resistor) for all inductors.
    - Note: $V_L(t < t_o) = 0V$.
- Calculate $V_C(t < t_o)$ and $I_L(t < t_o)$.

Final Condition

- Redraw the circuit at $t = \infty$ s
- Determine the value of all voltage and current sources at $t = \infty$ s
- Make the appropriate substitutions for the energy storage devices.
  - Substitute an open circuit ($\infty\Omega$ resistor) for all capacitors.
    - Note: $i_C(t = \infty\ s) = 0A$.
  - Substitute an short circuit (0Ω resistor) for all inductors.
    - Note: $v_L(t = \infty\ s) = 0V$.
- Calculate $v_C(t = \infty\ s)$ and $i_L(t = \infty\ s)$.

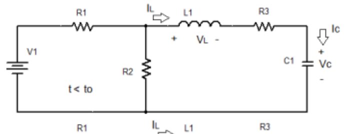
Example 01…

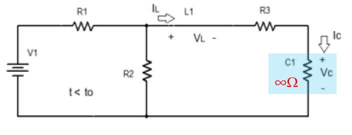
- The switch in the circuit has been closed for a long time. It is open at $t = t_0$.
  - Find the Boundary Conditions
    - i_L, v_L, i_C, v_C



…Example 01…

- At the initial condition the circuit is:



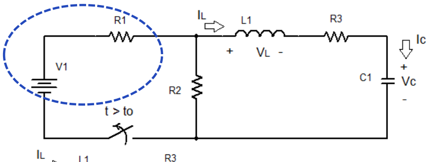


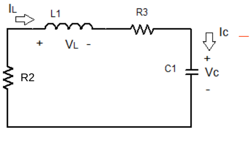
$i_L(-\infty) = i_L(t_o^-) = 0$ A $\quad v_L(-\infty) = v_L(t_o^-) = 0$ V

$i_C(-\infty) = i_C(t_o^-) = 0$ A $\quad v_C(-\infty) = v_C(t_o^-) = [R_2/(R_1+R_2)]V1$

…Example 01…

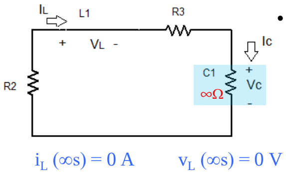
- At the final condition the switch opens,
  - which removes V1 and R1 from the circuit.





- The energy stored in the inductor and capacitor will be dissipated through R2 and R3 as t increased from $t = t_o$.

…Example 01



- At time $t = \infty s$, the energy stored in the inductor and in the capacitor will be completely released to the circuit.

$i_L(\infty s) = 0$ A $\quad v_L(\infty s) = 0$ V $\quad i_C(\infty s) = 0$ A $\quad v_C(\infty s) = 0$ V

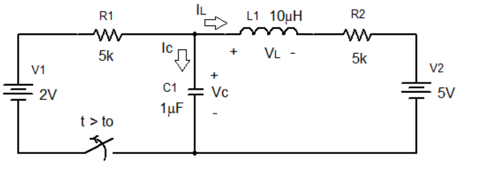
- For $t_o < t << \infty s$

$i_L(t) \neq 0$ A $\quad v_L(t) \neq 0$ V

$i_C(t) \neq 0$ A $\quad v_C(t) \neq 0$ V

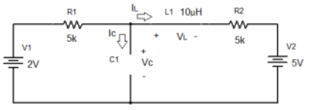
Example 02…

- The switch in the circuit has been closed for a long time. It is open at $t = t_0$.
  - Find the Boundary Conditions
    - i_L, v_L, i_C, v_C

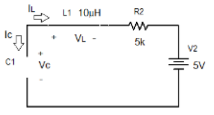


…Example 02

$t < t_o$



$t >> t_o$

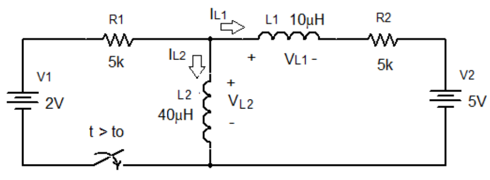


$i_L\ (-\infty s) = 0.3$ mA
$v_L\ (-\infty s) = 0$ V
$i_C\ (-\infty s) = 0$ A
$v_C\ (-\infty s) = 3.5$ V

$i_L\ (\infty s) = 0$ A
$v_L\ (\infty s) = 0$ V
$i_C\ (\infty s) = 0$ A
$v_C\ (\infty s) = 5$ V

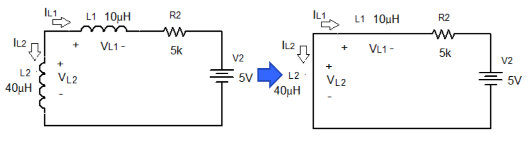
Example 03…

- The switch in the circuit has been open for a long time. It is open at $t = t_0$.
  - Find the Boundary Conditions
    - i_{L1}, v_{L1}, i_{L2}, v_{L2}



…Example 03…

$t < t_o$



$i_{L1}\ (-\infty s) = -1$ mA

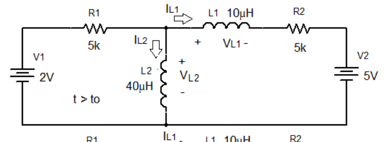
$v_{L1}\ (-\infty s) = 0$ V

$i_{L2}\ (-\infty s) = 1$ mA

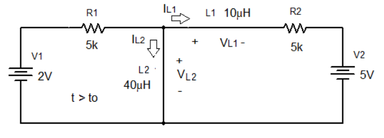
$v_{L1}\ (-\infty s) = 0$ V

…Example 03

$t > t_o$



$t >> t_o$

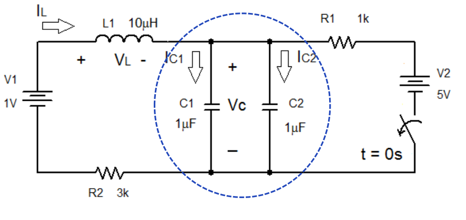


$i_{L1}\ (\infty s) = -1$ mA
$i_{L2}\ (\infty s) = 1.4$ mA

$v_{L1}\ (\infty s) = 0$ V
$v_{L2}(\infty s) = 0$ V

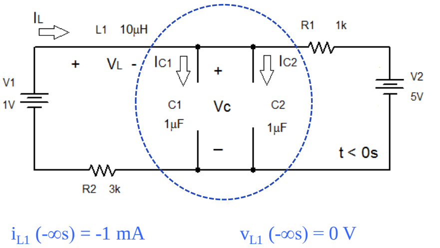
Example 04…

- The switch in the circuit has been closed for a long time. It is closed at $t = 0$.
  - Find the Boundary Conditions
    - i_L, v_L, i_{C1}, v_{C1}, i_{C2}, v_{C2}



…Example 04…

$t < 0$



$i_{L1}\ (-\infty s) = -1$ mA

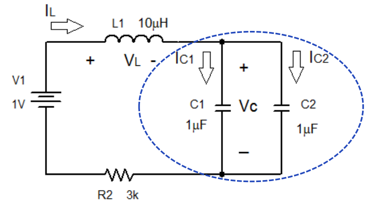
$v_{L1}\ (-\infty s) = 0$ V

$i_{C1}\ (-\infty s) = i_{C2}\ (-\infty s) = 0$ A

$v_{C1}\ (-\infty s) = v_{C2}\ (-\infty s) = 4$ V

…Example 04

$t >> 0$



$i_{L1}(\infty s) = 0mA$ $\quad v_{L1}(\infty s) = 0V$

$v_{C1}(\infty s) = v_{C2}(\infty s) = 1V$ $\quad i_{C1}(\infty s) = i_{C2}(\infty s) = 0A$

Summary

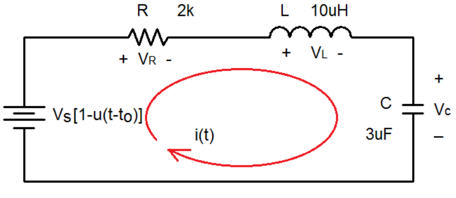
- Calculation of the initial and final conditions for 2nd order circuits requires:
  - Knowledge of the magnitude of the voltage and/or current sources in the circuit before and after a step function transition.
  - In steady state ($t < t_o$ and $t = \infty s$), replace energy storage devices.
    - Capacitors are opens circuits => $i_C = 0$ A
    - Inductors are short circuits => $v_L = 0$ V
  - Calculate the voltage across the capacitor and the current through the inductor.
- During the transition at the step $t = t_o$
  - Voltage across a capacitor is continuous
    - $v_C(t_o^+) = v_C(t_o^-)$
  - Current through an inductor is continuous
    - $i_L(t_o^+) = i_L(t_o^-)$

Source-Free RLC Circuit-Series RLC Network

- Objective of Lecture
  - Derive the equations that relate the voltages across and currents flowing through a resistor, an inductor, and a capacitor in series as:
    - the unit step function associated with voltage or current source changes from 1 to 0 or
    - a switch disconnects a voltage or current source into the circuit.
  - Describe the solution to the 2nd order equations when the condition is:
    - Overdamped
    - Critically Damped
    - Underdamped

Series RLC Network

- With a step function voltage source.



Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0 V at $t < t_o$
    - $i(t_o^-) = i_L(t_o^-) = 0$ A and $v_C(t_o^-) = Vs$
    - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i(\infty s) = i_L(\infty s) = 0$ A and $v_C(\infty s) = 0$ V
    - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A

Selection of Parameter

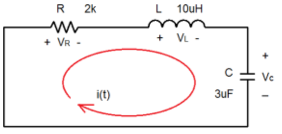
- Initial Conditions
  - $i(t_o^-) = i_L(t_o^-) = 0$ A and $v_C(t_o^-) = Vs$
  - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
- Final Conditions
  - $i(\infty s) = i_L(\infty s) = 0A$ and $v_C(\infty s) = 0$ V
  - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A
- Since the voltage across the capacitor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $v_C(t)$.

Kirchhoff's Voltage Law

$$\sum v(t) = 0$$
$$v_C(t) + L\frac{di_L(t)}{dt} + Ri_L = 0$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$
$$i_L(t) = i_C(t)$$
$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$
$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = 0$$

Second order Differential Equation

$$v_C(t-t_o) = v_t(t-t_o) \text{ when } t > t_o$$



General Solution…

Let $v_C(t) = Ae^{s\Delta t}$

$$As^2e^{s\Delta t} + \frac{AR}{L}se^{s\Delta t} + \frac{A}{LC}e^{s\Delta t} = 0$$
$$Ae^{s\Delta t}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

…General Solution…

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Characteristic Equation

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Roots of Characteristic Equation

…General Solution…

- A more compact way of expressing the roots:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad \alpha = \frac{R}{2L}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$
$$s^2 + 2\alpha s + \omega_o^2 = 0$$

- The roots s_1 and s_2 are called natural frequencies, measured in nepers per second (Np/s),
  - because they are associated with the natural response of the circuit;
- ω_0 is known as the resonant frequency or strictly as the undamped natural frequency, expressed in radians per second (rad/s);
- α is the neper frequency or the damping factor, expressed in nepers per second.

….General Solution

$$v_{C1}(t) = A_1e^{s_1\Delta t}$$
$$v_{C2}(t) = A_2e^{s_2\Delta t}$$
$$v_C(t) = v_{C1}(t) + v_{C2}(t) = A_1e^{s_1\Delta t} + A_2e^{s_2\Delta t}$$

Solve for Coefficients A_1 and A_2

- Use the boundary conditions at t_o^- and $t = \infty s$ to solve for A1 and A2.

$$v_C(t_o^-) = V_S$$

  - Since the voltage across a capacitor must be a continuous function of time.

$$v_C(t_o^-) = v_C(t_o^+) = v_{C1}(t_o^+) + v_{C2}(t_o^+) = V_S$$
$$A_1e^{s_1(0s)} + A_2e^{s_2(0s)} = A_1 + A_2 = V_S$$

  - Also know that

$$i_C(t_o) = C\frac{dv_C(t_o)}{dt} = \frac{d}{dt}\left[v_{C1}(t_o) + v_{C2}(t_o)\right] = 0$$
$$s_1A_1e^{s_1(0s)} + s_2A_2e^{s_2(0s)} = s_1A_1 + s_2A_2 = 0$$

Three types of solutions

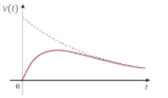
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- From these equations, we can infer that there are three types of solutions:
  - If $\alpha > \omega_0$, we have the overdamped case.
  - If $\alpha = \omega_0$, we have the critically damped case.
  - If $\alpha < \omega_0$, we have the underdamped case.
- We will consider each of these cases separately.

Three types of solutions

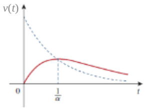
- Overdamped Case ($\alpha > \omega_o$)
  - implies that $C > 4L/R^2$
    - s_1 and s_2 are negative and real numbers

$$v_C(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t}$$



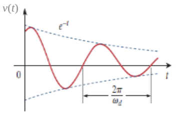
- Critically damped Case ($\alpha = \omega_o$)
  - implies that $C = 4L/R^2$
    - $s_1 = s_2 = -\alpha = -R/2L$

$$v_C(t) = A_1 e^{-\alpha \Delta t} + A_2 \Delta t e^{-\alpha \Delta t}$$



Three types of solutions

- Underdamped Case ($\alpha < \omega_o$)
  - implies that $C < 4L/R^2$



$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j\omega_d$$
$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

- $j = \sqrt{-1}$, i is used by the mathematicians for imaginary numbers

Three types of solutions

$$v_C(t) = e^{-\alpha \Delta t}(A_1 e^{j\omega_d \Delta t} + A_2 e^{-j\omega_d \Delta t})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

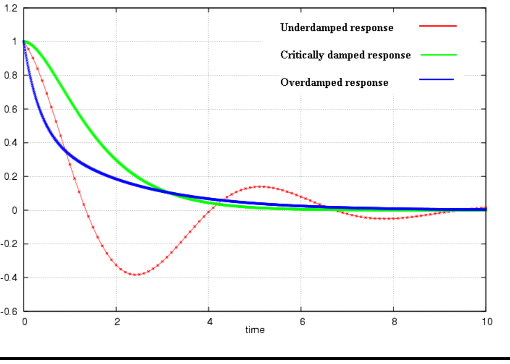
$$v_C(t) = e^{-\alpha \Delta t}[A_1(\cos\omega_d \Delta t + j\sin\omega_d \Delta t) + A_2(\cos\omega_d \Delta t - j\sin\omega_d \Delta t)]$$
$$v_C(t) = e^{-\alpha \Delta t}[(A_1 + A_2)\cos\omega_d \Delta t + j(A_1 - A_2)\sin\omega_d \Delta t]$$
$$v_C(t) = e^{-\alpha \Delta t}[B_1 \cos\omega_d \Delta t + B_2 \sin\omega_d \Delta t]$$

$$B_1 = A_1 + A_2 \qquad B_2 = j(A_1 - A_2)$$

Angular Frequencies

- ω_o is called the undamped natural frequency
  - The frequency at which the energy stored in the capacitor flows to the inductor and then flows back to the capacitor.
    - If $R = 0\Omega$, this will occur forever.
- ω_d is called the damped natural frequency
  - Since the resistance of R is not usually equal to zero, some energy will be dissipated through the resistor as energy is transferred between the inductor and capacitor.
    - α determined the rate of the damping response.

Three types of solutions



Properties of RLC network

- Behavior of RLC network is described as damping, which is a gradual loss of the initial stored energy
  - The resistor R causes the loss
  - α determined the rate of the damping response
    - If R = 0, the circuit is loss-less and energy is shifted back and forth between the inductor and capacitor forever at the natural frequency.

Properties of RLC network

- Oscillatory response of a lossy RLC network is possible because the energy in the inductor and capacitor can be transferred from one component to the other.
  - Underdamped response is a damped oscillation, which is called ringing.
- Critically damped circuits reach the final steady state in the shortest amount of time as compared to overdamped and underdamped circuits.
  - However, the initial change of an overdamped or underdamped circuit may be greater than that obtained using a critically damped circuit.

Set of Solutions when t > t_o

- There are three different solutions which depend on the magnitudes of the coefficients of the $\frac{dv_C(t)}{dt}$ and the $v_C(t)$ terms.
  - To determine which one to use, you need to calculate the natural angular frequency of the series RLC network and the term α.

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{R}{2L}$$

Transient Solutions when t > t_o

- Overdamped response ($\alpha > \omega_o$)
  where $\Delta t = t - t_o$

$$\boxed{\begin{aligned} v_C(t) &= A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t} \\ s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}}$$

- Critically damped response ($\alpha = \omega_o$)

$$\boxed{v_C(t) = (A_1 + A_2 \Delta t) e^{-\alpha \Delta t}}$$

- Underdamped response ($\alpha < \omega_o$)

$$\boxed{\begin{aligned} & v_C(t) = [B_1 \cos(\omega_d \Delta t) + B_2 \sin(\omega_d \Delta t)] e^{-\alpha \Delta t} \\ & \omega_d = \sqrt{\omega_o{}^2 - \alpha^2}\,, \quad B_1 = A_1 + A_2\,, \quad B_2 = j(A_1 - A_2) \end{aligned}}$$

Find Coefficients

- After you have selected the form for the solution based upon the values of ω_o and α
  - Solve for the coefficients in the equation by evaluating the equation at $t = t_o^-$ and $t = \infty s$ using the initial and final boundary conditions for the voltage across the capacitor.
    - $v_C(t_o^-) = Vs$
    - $v_C(\infty s) = 0$ V

Other Voltages and Currents

- Once the voltage across the capacitor is known, the following equations for the case where $t > t_o$ can be used to find:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i(t) = i_C(t) = i_L(t) = i_R(t)$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$v_R(t) = Ri_R(t)$$

Solutions when $t < t_o$

- The initial conditions of all of the components are the solutions for all times $-\infty s < t < t_o$.
  - $v_C(t) = Vs$
  - $i_C(t) = 0$ A
  - $v_L(t) = 0$ V
  - $i_L(t) = 0$ A
  - $v_R(t) = 0$ V
  - $i_R(t) = 0$ A

Summary

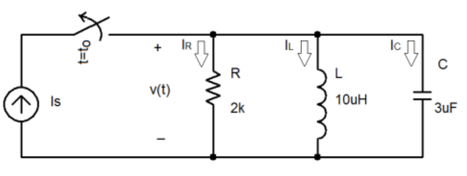
- The set of solutions when $t > t_o$ for the voltage across the capacitor in a RLC network in series was obtained.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty s$.
  - The voltage across the capacitor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the current through the capacitor and the voltages and currents for the inductor and resistor can be calculated.

Source-Free RLC Circuit-Parallel RLC Network

- Objective of Lecture
  - Derive the equations that relate the voltages across and currents flowing through a resistor, an inductor, and a capacitor in parallel as:
    - the unit step function associated with voltage or current source changes from 1 to 0 or
    - a switch disconnects a voltage or current source into the circuit.
  - Describe the solution to the 2^{nd} order equations when the condition is:
    - Overdamped
    - Critically Damped
    - Underdamped

RLC Network

- A parallel RLC network where the current source is switched out of the circuit at $t = t_o$.



Boundary Conditions

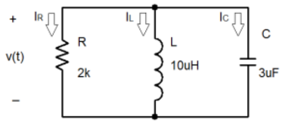
- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0V at $t < t_o$
    - $i_L(t_o^-) = Is$ and $v(t_o^-) = v_C(t_o^-) = 0$ V
    - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0$ A
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i_L(\infty s) = 0A$ and $v(\infty s) = v_C(\infty s) = 0$ V
    - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0$ A

Selection of Parameter

- Initial Conditions
  - $i_L(t_o^-) = Is$ and $v(t_o^-) = v_C(t_o^-) = 0$ V
  - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0$ A
- Final Conditions
  - $i_L(\infty s) = 0A$ and $v(\infty s) = v_C(\infty s) = 0$ V
  - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0$ A
- Since the current through the inductor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $i_L(t)$.

Kirchoff's Current Law

$$i_R(t)+i_L(t)+i_C(t)=0$$
$$v(t)=v_R(t)=v_L(t)=v_C(t)$$
$$\frac{v_R(t)}{R}+i_L(t)+C\frac{dv_C(t)}{dt}=0$$
$$v_L(t)=v(t)=L\frac{di_L(t)}{dt}$$
$$LC\frac{d^2i_L(t)}{dt^2}+\frac{L}{R}\frac{di_L(t)}{dt}+i_L(t)=0$$
$$\frac{d^2i_L(t)}{dt^2}+\frac{1}{RC}\frac{di_L(t)}{dt}+\frac{i_L(t)}{LC}=0$$



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General Solution

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

$$s_1=-\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$
$$s_2=-\frac{1}{2RC}-\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

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$$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^2} \qquad \alpha=\frac{1}{2RC}$$
$$s_2=-\alpha-\sqrt{\alpha^2-\omega_o^2} \qquad \omega_o=\frac{1}{\sqrt{LC}}$$

$$s^2+2\alpha s+\omega_o^2=0$$

Note that the equation for the natural frequency of the RLC circuit is the same whether the components are in series or in parallel.

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Three types of solutions

- Overdamped Case ($\alpha > \omega_o$)
  - implies that $L > 4R^2C$
    - s_1 and s_2 are negative and real numbers

$$i_{L1}(t)=A_1e^{s_1\Delta t}$$
$$i_{L2}(t)=A_2e^{s_2\Delta t}$$
$$\Delta t=t-t_o$$
$$i_L(t)=i_{L1}(t)+i_{L2}(t)=A_1e^{s_1\Delta t}+A_2e^{s_2\Delta t}$$

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Three types of solutions

- Critically Damped Case ($\alpha = \omega_o$)
  - implies that $L = 4R^2C$
    - $s_1 = s_2 = -\alpha = -1/2RC$

$$i_L(t)=A_1e^{-\alpha\Delta t}+A_2\Delta te^{-\alpha\Delta t}$$

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Three types of solutions

- Underdamped Case ($\alpha < \omega_o$)
  - implies that $L < 4R^2C$

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^2}=-\alpha+j\omega_d$$
$$s_2=-\alpha-\sqrt{\alpha^2-\omega_o^2}=-\alpha-j\omega_d$$
$$\omega_d=\sqrt{\omega_o^2-\alpha^2}$$
$$i_L(t)=e^{-\alpha\Delta t}[B_1\cos\omega_d\Delta t+B_2\sin\omega_d\Delta t]$$
$$B_1=A_1+A_2\,,\ B_2=j(A_1-A_2)$$

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Other Voltages and Currents

- Once current through the inductor is known:

$$v_L(t) = L\frac{di_L(t)}{dt}$$
$$v_L(t) = v_C(t) = v_R(t)$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$
$$i_R(t) = v_R(t)/R$$

Summary

- The set of solutions when $t > t_o$ for the current through the inductor in a RLC network in parallel was obtained.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty s$.
  - The current through the inductor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the voltage across the inductor and the voltages and currents for the capacitor and resistor can be calculated.

Summary of Relevant Equations for Source-Free RLC Circuits

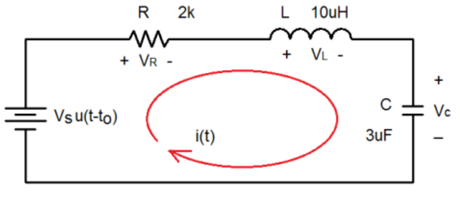
Type	Condition	Criteria	α	ω_0	Response
Parallel	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$A_1e^{s_1t} + A_2e^{s_2t}$, where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$
Series			$\frac{R}{2L}$		
Parallel	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1t + A_2)$
Series			$\frac{R}{2L}$		
Parallel	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$; $B_1 = A_1 + A_2 \quad B_2 = j(A_1 - A_2)$
Series			$\frac{R}{2L}$		

Step Response-Series RLC Network

- Objective of Lecture
  - Derive the equations that relate the voltages across a resistor, an inductor, and a capacitor in series as:
    - the unit step function associated with voltage or current source changes from 0 to 1 or
    - a switch connects a voltage or current source into the circuit.
  - Describe the solution to the 2nd order equations when the condition is:
    - Overdamped
    - Critically Damped
    - Underdamped

Series RLC Network

- With a step function voltage source.



Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0V at $t < t_o$
    - $i(t_o^-) = i_L(t_o^-) = 0$ A and $v_C(t_o^-) = 0$ V
    - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i(\infty s) = i_L(\infty s) = 0$ A and $v_C(\infty s) = V_s$
    - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A

Selection of Parameter

- Initial Conditions
  - $i(t_o^-) = i_L(t_o^-) = 0$ A and $v_C(t_o^-) = 0$ V
  - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
- Final Conditions
  - $i(\infty s) = i_L(\infty s) = 0$ A and $v_C(\infty s) = Vs$
  - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A
- Since the voltage across the capacitor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $v_C(t)$.

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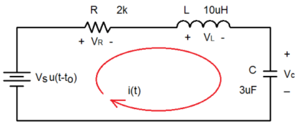
Kirchhoff's Voltage Law

$$\sum v(t) = 0$$
$$v_C(t) + L\frac{di_L(t)}{dt} + Ri_L - V_S = 0$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$
$$i_L(t) = i_C(t)$$
$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = V_S$$
$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_S}{LC}$$
$$v_C(t-t_o) = v_t(t-t_o) + v_{ss}(t-t_o) \text{ when } t > t_o$$



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Set of Solutions when t > t_o

- Similar to the solutions for the natural response, there are three different solutions.
- To determine which one to use, you need to calculate the natural angular frequency of the series RLC network and the term α.

$$\omega_o = \frac{1}{\sqrt{LC}}$$
$$\alpha = \frac{R}{2L}$$

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Transient Solutions when t > t_o

- Overdamped response ($\alpha > \omega_o$)
  where $t - t_o = \Delta t$

$$\boxed{\begin{aligned} v_C(t) &= A_1e^{s_1\Delta t} + A_2e^{s_2\Delta t} \\ s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}}$$

- Critically damped response ($\alpha = \omega_o$)

$$\boxed{v_C(t) = (A_1 + A_2\Delta t)e^{-\alpha\Delta t}}$$

- Underdamped response ($\alpha < \omega_o$)

$$\boxed{\begin{aligned} &v_C(t) = [B_1\cos(\omega_d\Delta t) + B_2\sin(\omega_d\Delta t)]e^{-\alpha\Delta t} \\ &\omega_d = \sqrt{\omega_o^2 - \alpha^2}\ ,\quad B_1 = A_1 + A_2\ ,\quad B_2 = j(A_1 - A_2) \end{aligned}}$$

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Steady State Solutions when t > t_o

- The final condition of the voltages across the capacitor is the steady state solution.
  - $v_C(\infty s) = Vs$

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Complete Solution when t > t_o

- Overdamped response

$$v_C(t) = A_1e^{s_1\Delta t} + A_2e^{s_2\Delta t} + Vs$$

- Critically damped response

$$v_C(t) = (A_1 + A_2\Delta t)e^{-\alpha\Delta t} + Vs$$

- Underdamped response

$$v_C(t) = [B_1\cos(\omega_d\Delta t) + B_2\sin(\omega_d\Delta t)]e^{-\alpha\Delta t} + Vs$$
$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}\ ,\quad B_1 = A_1 + A_2\ ,\quad B_2 = j(A_1 - A_2)$$

where $\Delta t = t - t_o$

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Other Voltages and Currents

- Once the voltage across the capacitor is known, the following equations for the case where t > t_o can be used to find:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$
$$i(t) = i_C(t) = i_L(t) = i_R(t)$$
$$v_L(t) = L\frac{di_L(t)}{dt}$$
$$v_R(t) = Ri_R(t)$$

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Summary

- The set of solutions when $t > t_o$ for the voltage across the capacitor in a RLC network in series was obtained.
  - The final condition for the voltage across the capacitor is the steady state solution.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty s$.
  - The voltage across the capacitor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the current through the capacitor and the voltages and currents for the inductor and resistor can be calculated.

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Step Response-Parallel RLC Network

- Objective of Lecture
  - Derive the equations that relate the voltages across a resistor, an inductor, and a capacitor in parallel as:
    - the unit step function associated with voltage or current source changes from 0 to 1 or
    - a switch connects a voltage or current source into the circuit.
  - Describe the solution to the 2nd order equations when the condition is:
    - Overdamped
    - Critically Damped
    - Underdamped

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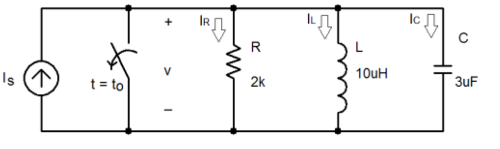
Parallel RLC Network

- With a current source switched into the circuit at $t = t_o$.



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Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$.
  - Since the voltage source has a magnitude of 0V at $t < t_o$
    - $i_L(t_o^-) = 0$ A and $v(t_o^-) = v_C(t_o^-) = 0$ V
    - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
  - Once the steady state is reached after the voltage source has a magnitude of Vs at $t > t_o$, replace the capacitor with an open circuit and the inductor with a short circuit.
    - $i_L(\infty s) = Is$ and $v(\infty s) = v_C(\infty s) = 0$ V
    - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A

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Selection of Parameter

- Initial Conditions
  - $i_L(t_o^-) = 0$ A and $v(t_o^-) = v_C(t_o^-) = 0$ V
  - $v_L(t_o^-) = 0$ V and $i_C(t_o^-) = 0$ A
- Final Conditions
  - $i_L(\infty s) = Is$ and $v(\infty s) = v_C(\infty s) = 0$ V
  - $v_L(\infty s) = 0$ V and $i_C(\infty s) = 0$ A

- Since the current through the inductor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $i_L(t)$.

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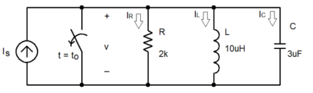
Kirchhoff's Current Law

$$i_R(t)+i_L(t)+i_C(t)=i_S(t)$$
$$v(t)=v_R(t)=v_L(t)=v_C(t)$$
$$\frac{v_R(t)}{R}+i_L(t)+C\frac{dv_C(t)}{dt}=I_S$$
$$v_L(t)=v(t)=L\frac{di_L(t)}{dt}$$
$$LC\frac{d^2i_L(t)}{dt^2}+\frac{L}{R}\frac{di_L(t)}{dt}+i_L(t)=I_S$$
$$\frac{d^2i_L(t)}{dt^2}+\frac{1}{RC}\frac{di_L(t)}{dt}+\frac{i_L(t)}{LC}=\frac{I_S}{LC}$$



$$i_L(t)=i_t(t)+i_{ss}(t)$$

Set of Solutions when t > t_o

- Similar to the solutions for the natural response, there are three different solutions.
- To determine which one to use, you need to calculate the natural angular frequency of the parallel RLC network and the term α.

$$\omega_o=\frac{1}{\sqrt{LC}}$$
$$\alpha=\frac{1}{2RC}$$

Transient Solutions when t > t_o

- Overdamped response

$$\boxed{i_L(t)=A_1e^{s_1\Delta t}+A_2e^{s_2\Delta t}}$$

- Critically damped response

$$\boxed{i_L(t)=(A_1+A_2\Delta t)e^{-\alpha\Delta t}}$$

- Underdamped response

$$\boxed{\begin{array}{l} i_L(t)=[B_1\cos(\omega_d\Delta t)+B_2\sin(\omega_d\Delta t)]e^{-\alpha\Delta t} \\ \omega_d=\sqrt{\omega_o{}^2-\alpha^2}\;,\quad B_1=A_1+A_2\;,\quad B_2=j(A_1-A_2) \end{array}}$$

where $\Delta t=t-t_o$

Other Voltages and Currents

- Once the current through the inductor is known:

$$v_L(t)=L\frac{di_L(t)}{dt}$$
$$v_L(t)=v_C(t)=v_R(t)$$
$$i_C(t)=C\frac{dv_C(t)}{dt}$$
$$i_R(t)=v_R(t)/R$$

Complete Solution when t > t_o

- Overdamped response

$$i_L(t)=A_1e^{s_1\Delta t}+A_2e^{s_2\Delta t}+Is$$

- Critically damped response

$$i_L(t)=(A_1+A_2\Delta t)e^{-\alpha\Delta t}+Is$$

- Underdamped response

$$i_L(t)=[B_1\cos(\omega_d\Delta t)+B_2\sin(\omega_d\Delta t)]e^{-\alpha\Delta t}+Is$$
$$\omega_d=\sqrt{\omega_o{}^2-\alpha^2}\;,\quad B_1=A_1+A_2\;,\quad B_2=j(A_1-A_2)$$

Summary

- The set of solutions when $t > t_o$ for the current through the inductor in a RLC network in parallel was obtained.
  - The final condition for the current through the inductor is the steady state solution.
  - Selection of equations is determine by comparing the natural frequency ω_o to α.
  - Coefficients are found by evaluating the equation and its first derivation at $t = t_o^-$ and $t = \infty s$.
  - The current through the inductor is equal to the initial condition when $t < t_o$
- Using the relationships between current and voltage, the voltage across the inductor and the voltages and currents for the capacitor and resistor can be calculated.

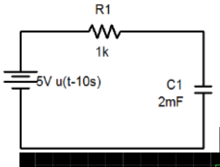
Duality

- Objective of Lecture
  - Introduce the concept of duality.

Parallelism Between Components

- Two circuits are said to be duals of one another if they are described by the same characterizing equations with the dual pairs interchanged.

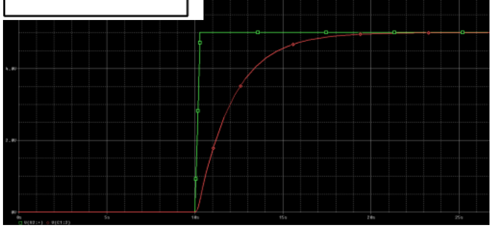
Dual Pairs	
Resistance (R)	**Conductance (G)**
Inductance (L)	**Capacitance (C)**
Voltage (v)	**Current (i)**
Voltage Source	**Current Source**
Node	**Mesh/Loop**
Series Path	**Parallel Path**
Open Circuit	**Short Circuit**
KVL	**KCL**
Thévenin	**Norton**

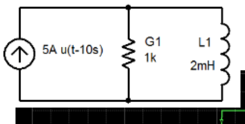


$V_{C1}(t) = 5V[1-e^{-(t-10s)/\tau}]$

$\tau = RC = 2s$

$V_{C1}(t) = 5V[1-e^{-(t-10s)/2s}]$





$I_{L1}(t) = 5A[1-e^{-(t-10s)/\tau}]$

$\tau = L1/R1 = L1G1$

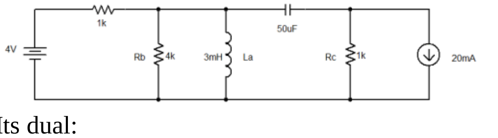
$I_{L1}(t) = 5A[1-e^{-(t-10s)/2s}]$

To Construct Dual Circuits

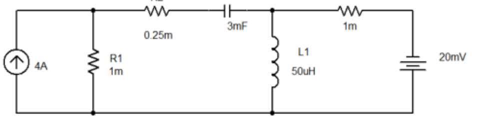
- Place a node at the center of each mesh of the circuit.
- Place a reference node (ground) outside of the circuit.
- Draw lines between nodes such that each line crosses an element.
- Replace the element by its dual pair.
- Determine the polarity of the voltage source and direction of the current source.
  - A voltage source that produces a positive mesh current has as its dual a current source that forces current to flow from the reference ground to the node associated with that mesh.

Example 05

- Circuit:



- Its dual:



Summary

- The principle of duality means that the solution to one circuit can be applied to multiple other circuits that can be described using the same set of equations in which the variables have been interchanged.