Advanced Digital Signal Processing

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- **Time-Scale Analysis**
  - Wavelet Transform
  - Complex Wavelet

Introduction

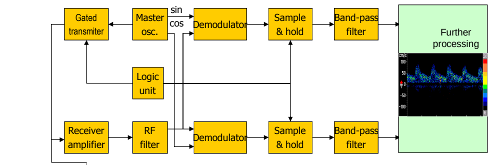
- Stroke is an illness causing partial or total paralysis, or death.
- The most common type of stroke (80% of all strokes) occurs when a blood vessel in or around the brain becomes plugged.
- The plug can originate in an artery of the brain or somewhere else in the body, often the heart, where it breaks off and travels up the arterial tree to the brain, until it lodges in a blood vessel.

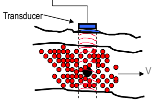
Emboli

- These "travelling clots" are called emboli.
- Solid emboli typically consist of
  - thrombus,
  - hard calcified plaque or
  - soft fatty atheroma.
- Gaseous emboli may also enter the circulation during surgery or form internally from gases that are normally dissolved in the blood.
- Any foreign body (solid or gas) that becomes free-floating in the bloodstream is called an embolus, from the Greek 'embolos' meaning 'a stopper'.

- Early and accurate detection of asymptomatic emboli is important in identifying patients at high risk of stroke
- They can be detected by Doppler ultrasound
  - Transcranial Doppler ultrasound (TCD)
    - 1-2 MHz

Typical Doppler System for Detecting Emboli



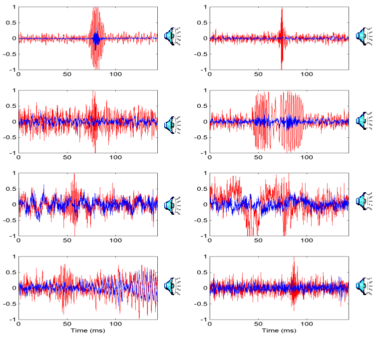


- **Demodulation**
- **Quadrature to directional signal conversion**
- **Time-frequency/scale analysis**
- **Data visualization**
- **Detection and estimation**
- **Derivation of diagnostic information**

Embolic Doppler ultrasound signal

- Within audio range (0-10 kHz)
- Appear as increasing and then decreasing in intensity for a short duration,
  - usually less than 300 ms.
- The bandwidth of ES is usually much less than that of Doppler Speckle.
  - narrow-band signals
- They are also oscillating and finite signals
  - Similar to wavelets.
- Have an associated characteristic click or chirping sound
- Unidirectional and usually contained within the flow spectrum
- The spectral content of an ES is also time dependent.

Examples of Embolic Signals



(red: forward, blue: reverse)

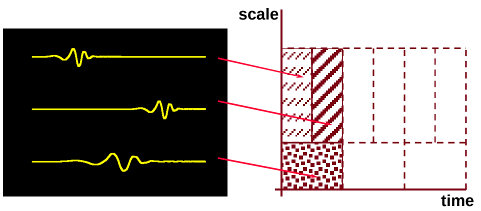
More examples

Time-Scale Analysis (Continuous Wavelet Transform)

$$W_s(a,b)=\frac{1}{\sqrt{|a|}}\int_{-\infty}^{+\infty} s(t)\,\psi^*\left(\frac{t-b}{a}\right)dt$$

- $\psi^*(t)$ is the analysing wavelet. a (>0) controls the scale of the wavelet. b is the translation and controls the position of the wavelet.
- Can be computed directly by convolving the signal with a scaled and dilated version of the wavelet (Frequency domain implementation may increase computational efficiency).
- Wavelets are ideally suited for the analysis of sudden short duration signal changes (non-stationary signals).
- Decomposes a time series into time-scale space, creating a three dimensional representation (time, wavelet scale and amplitude of the WT coefficients).

Wavelet Analysis

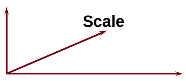


Non-stationary and multiscale signal analysis

- Good time resolution at high frequencies
- Good frequency resolution at low frequencies

Continuous Wavelet Transform

for each Scale
    for each Position
        Coefficient (S,P) = $\int_{\text{all time}}$ Signal × Wavelet (S,P)
    end
end



Wavelet analysis and synthesis

$$W_s(a,b) = |a|^{-1/2}\int_{-\infty}^{+\infty} s(t)\psi\left(\frac{t-b}{a}\right)dt, \quad s(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_s(a,b)|a|^{-1/2}\psi\left(\frac{t-b}{a}\right)\frac{dadb}{a^2}$$

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\Psi(\omega)|}{\omega}d\omega \quad < \infty$$

- A wavelet must oscillate and decay
- a (>0) controls the scale of the wavelet. b is the translation parameter and controls the position of the wavelet.

Time-frequency and Time-scale analysis

$$F_s(t,\omega) = \int_{-\infty}^{+\infty} s(t)g^*(t-\tau)e^{-j\omega t}dt$$

$$W_s(a,b) = \int_{-\infty}^{+\infty} s(t)\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)dt$$

TF tiling is fixed.

Assumes the signal is stationary within the analysis window.

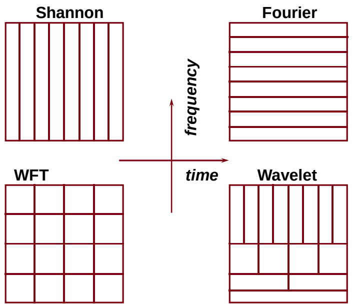
The FT has an inherent TF resolution limitation.

TS tiling is logarithmic.

Ideally suited for the analysis of sudden short duration signal changes.

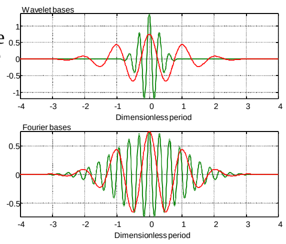
Allows TF resolution compromise to be optimised.

Time-Frequency Discretization



Difference between Wavelet and Fourier bases

- The basic difference between the WT and the WFT is that when the scale factor ***a*** is changed, the duration and the bandwidth of the wavelet are both changed but its shape remains the same.
- The WT uses short windows at high frequencies and long windows at low frequencies in contrast to the WFT, which uses a single analysis window.
- This partially overcomes the time resolution limitation of the WFT.



Complex Wavelet Transform

- Similar to complex FT, a complex WT providing directional information in scale domain exists.
- A complex wavelet $\psi(t)$ must also satisfy the following property:

$$\Psi(\omega) = \Psi^+ + \Psi^-, \qquad \Psi(\omega) = \begin{cases}\Psi^+ & if \quad a > 0 \\ \Psi^- & if \quad a < 0\end{cases}$$

- Morlet and Cauchy wavelets are two examples for such complex wavelets.

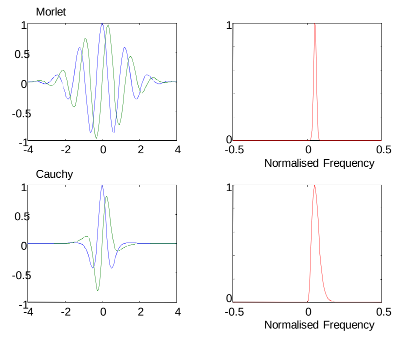
- Complex Morlet wavelet.

$$\psi(t/a) = e^{i\omega_0 t/a}\, e^{-t^2/2a^2}, \qquad \Psi(\omega) = \begin{cases}|a|\sqrt{2\pi}e^{\frac{-(\omega_0 - a\omega)^2}{2}}\mathrm{H}(\omega), & a > 0 \\ |a|\sqrt{2\pi}e^{\frac{-(\omega_0 + a\omega)^2}{2}}\mathrm{H}(-\omega), & a < 0\end{cases}$$

- Complex Cauchy wavelet.

$$\psi(t/a) = \frac{\Gamma(m+1)}{2\pi}(1 - i\frac{t}{a})^{-(m+1)}, \qquad \Psi(\omega) = \begin{cases}a^{m+1}\omega^m e^{-a\omega}\mathrm{H}(\omega), & a > 0 \\ a^{m+1}(-\omega)^m e^{a\omega}\mathrm{H}(-\omega), & a < 0\end{cases}$$

$$\Gamma(m) = \int_0^{+\infty} t^{m-1}e^{-t}dt \quad , \qquad \mathrm{H}(\omega) = \begin{cases}1, & \omega > 0 \\ 0, & \omega < 0\end{cases}$$



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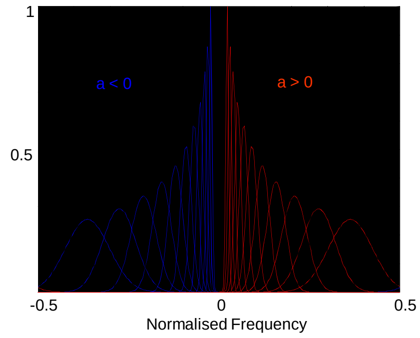

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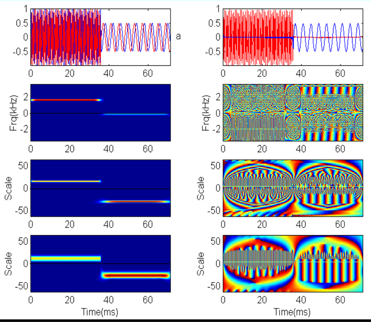
Time-frequency/scale analysis of Doppler Ultrasound Signals

$s(t) = e^{i2\pi f_f t}[u(t) - u(t-t_1)] + 0.5e^{-i2\pi f_r t}[u(t-t_1) - u(t-t_2)]$

$F_s = 7150$ Hz, $f_f = 1500$ Hz, $f_r = 250$ Hz, $t_1 = 35.8$ ms



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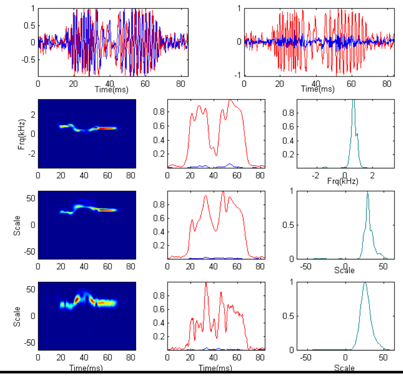
Example: analysis of Embolic Doppler signal



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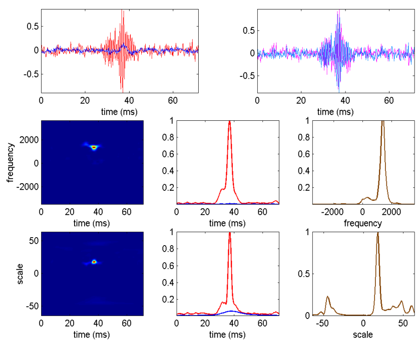
...CWT of an embolic Doppler signal...



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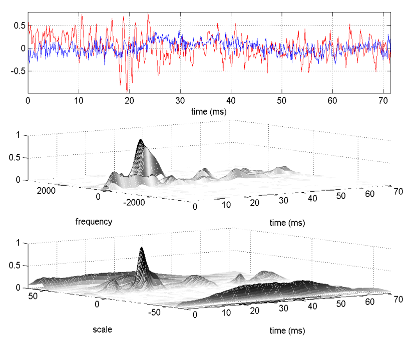
...CWT of an embolic Doppler signal



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Discrete Wavelet Transform

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Discrete Wavelet Transform (DWT)

$$W_s(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} s(t)\,\psi^*\left(\frac{t-b}{a}\right)dt$$

- Discrete WT is a special case of the continuous WT when $a=a_0j$ and $b=n.a_0j$.
- Dyadic wavelet bases are obtained when $a_0=2$

$$W_s(m,n) = \frac{1}{\sqrt{a_0^m}} \sum_{k=0}^{N-1} s(k)\,\psi\left(\frac{k-nb_0a_0^m}{a_0^m}\right)$$

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Discrete Wavelet Transform

- Scaling and positions are dyadic
- Fast algorithms exist
- Scaling function and wavelet must satisfy the following conditions

$$\int \phi(t)dt = 1, \quad \int \psi(t)dt = 0$$

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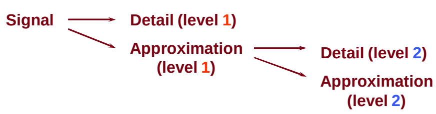
Discrete Wavelet Transform

- CWT: Scales = any value
- DWT: Scales = Dyadic scale $2^1\ 2^2\ 2^3\ 2^4\ 2^5$ ...
- Equivalent to:



$$S = D_1 + A_1 = D_1 + D_2 + A_2$$

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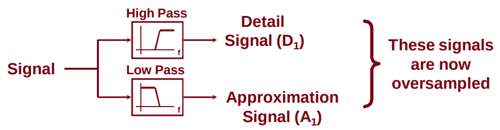
Signal Decomposition



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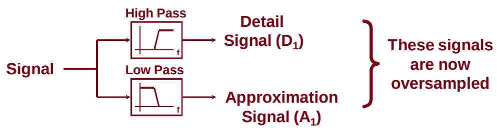
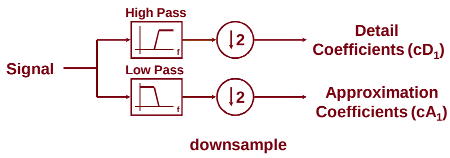
Signal Decomposition





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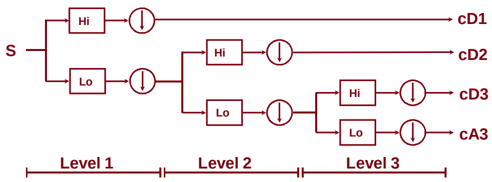
Decomposition Tree



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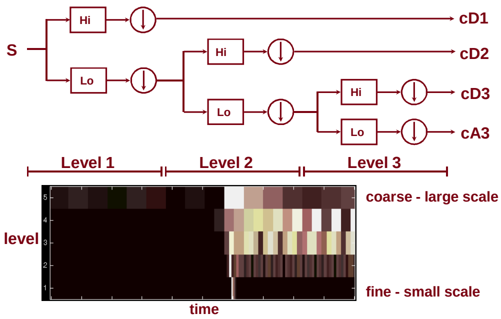
Decomposition Tree



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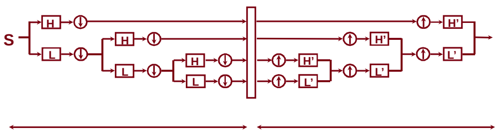
Analysis and Synthesis



- Decomposition
- Analysis
- DWT

- Reconstruction
- Synthesis
- IDWT

If we choose filters H, L, H' and L' correctly
⟶ PERFECT RECONSTRUCTION

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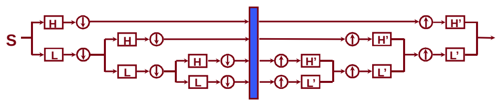
Compression and Denoising



- Compression:
  - Find the *minimum* number of coefficients so that the reconstructed signal still looks *reasonable*

- Denoising:
  Remove some coefficients so that the reconstructed signal has less noise

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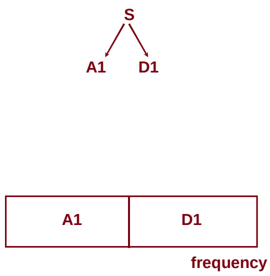
Wavelets...

Wavelet
Decomposition Tree



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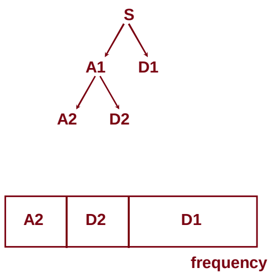
Wavelets Continued

Wavelet
Decomposition Tree



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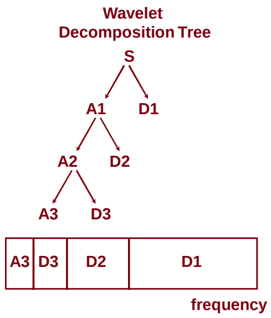
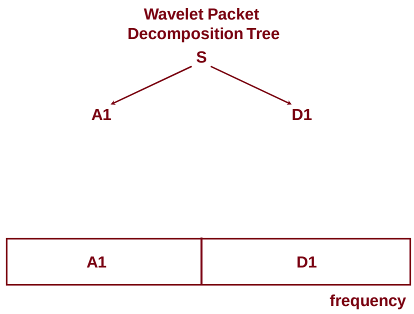
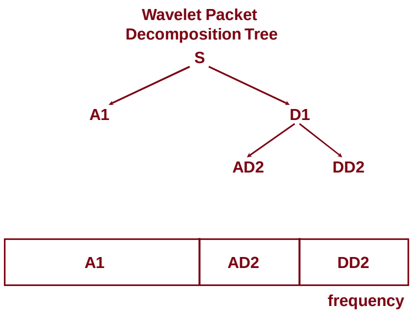
Wavelets Continued



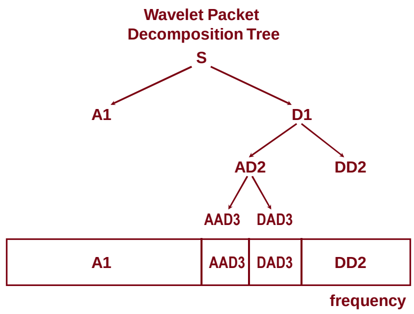
Wavelet Packets



Wavelet Packets



Wavelet Packets



Wavelet Packets