Advanced Digital Signal Processing

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Amplitude Modulation

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- Review of FT properties
  - Convolution <--> multiplication
  - Frequency shifting
- Sinewave Amplitude Modulation
  - AM radio
- Frequency-division multiplexing
  - FDM

Table of Easy FT Properties

Linearity Property

$$ax_1(t)+bx_2(t) \Leftrightarrow aX_1(j\omega)+bX_2(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d}X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|}X(j(\frac{\omega}{a}))$$

Table of FT Properties

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi}X(j\omega)*P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Frequency Shifting Property

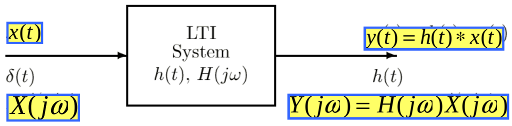
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{\infty} e^{-j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

$$= X(j(\omega-\omega_0))$$

$$y(t)=\frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega)=\begin{cases}1 & \omega_0-7<\omega<\omega_0+7 \\ 0 & elsewhere\end{cases}$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

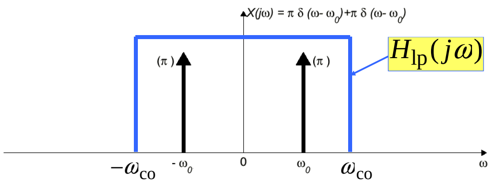
corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$= H(j\omega)[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$$
$$= H(j\omega_0)\pi\delta(\omega-\omega_0) + H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

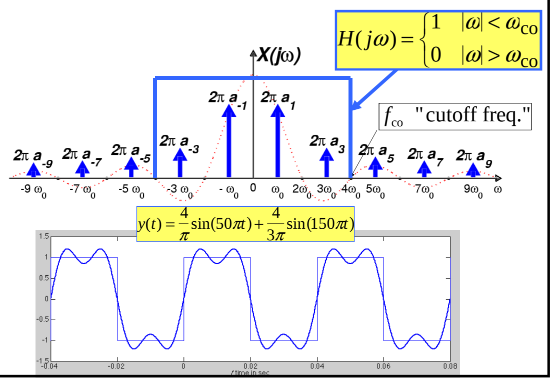
$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter



$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$
$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

Ideal LPF: Fourier Series



The way communication systems work

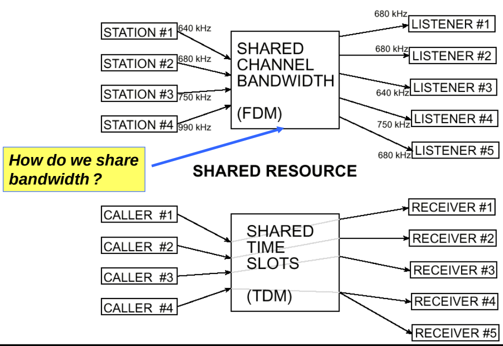


Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

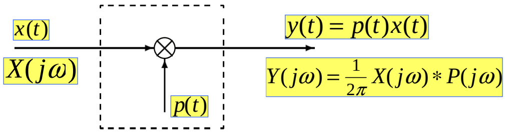
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

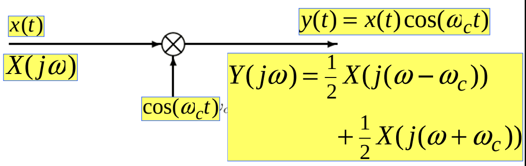
$$y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \tfrac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$p(t) = \cos(\omega_c t) \Leftrightarrow$$
$$P(j\omega) = \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$$

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$
$$Y(j\omega) = \tfrac{1}{2\pi} X(j\omega) * [\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)]$$

$$Y(j\omega) = \tfrac{1}{2} X(j(\omega-\omega_c)) + \tfrac{1}{2} X(j(\omega+\omega_c))$$

Amplitude Modulator



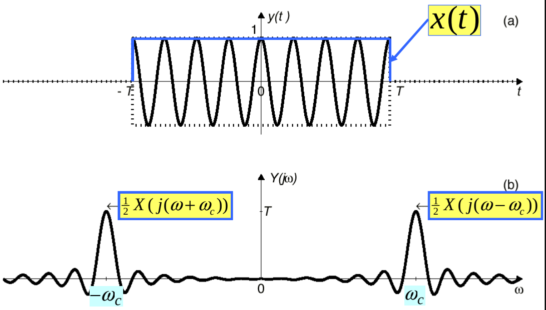
- *x(t)* modulates the amplitude of the cosine wave. The result in the frequency-domain is two shifted copies of $X(j\omega)$.

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$
$$Y(j\omega) = \tfrac{1}{2} X(j(\omega-\omega_c)) + \tfrac{1}{2} X(j(\omega+\omega_c))$$

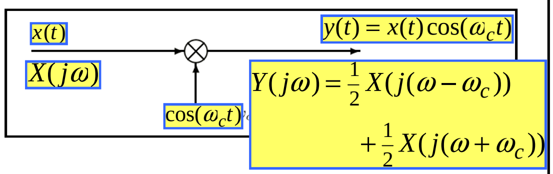
$$x(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases} \Leftrightarrow X(j\omega) = 2\frac{\sin(\omega T)}{(\omega)}$$

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$
$$Y(j\omega) = \frac{\sin((\omega-\omega_c)T)}{(\omega-\omega_c)} + \frac{\sin((\omega+\omega_c)T)}{(\omega+\omega_c)}$$

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$
$$Y(j\omega) = \tfrac{1}{2} X(j(\omega-\omega_c)) + \tfrac{1}{2} X(j(\omega+\omega_c))$$



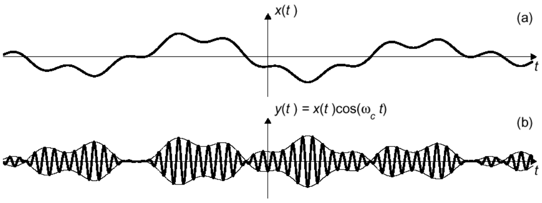
DSBAM Modulator



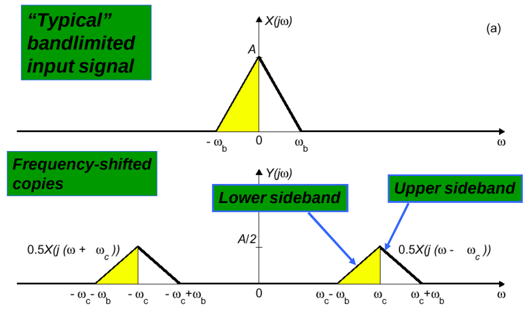
- If $X(j\omega)=0$ for $|\omega|>\omega_b$ and $\omega_c > \omega_b$, the result in the frequency-domain is two shifted and scaled **exact copies** of $X(j\omega)$.

DSBAM Waveform

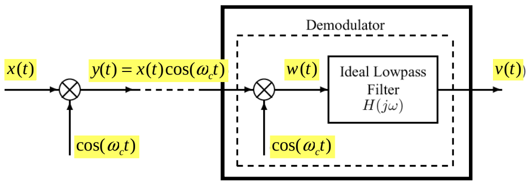
- In the time-domain, the "envelope" of sine-wave peaks follows |*x(t)*|



Double Sideband AM (DSBAM)



DSBAM DEmodulator

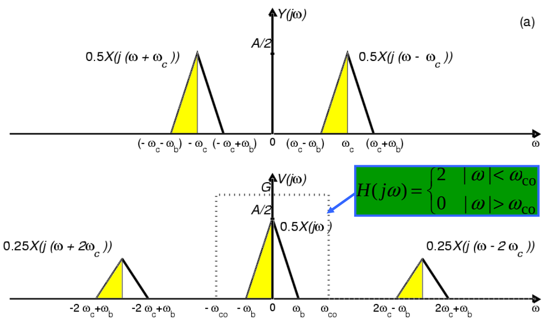


$$w(t) = x(t)[\cos(\omega_c t)]^2 = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_c t)$$

$$W(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega - 2\omega_c)) + \frac{1}{4}X(j(\omega + 2\omega_c))$$

$$V(j\omega) = H(j\omega)W(j\omega)$$

DSBAM Demodulation

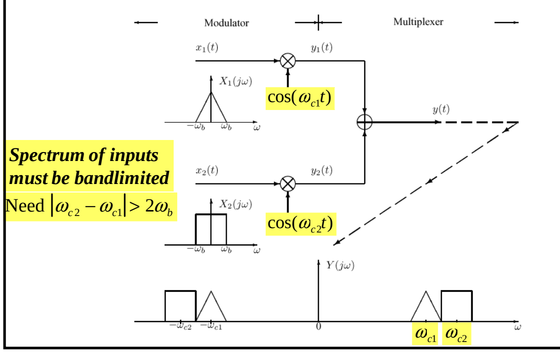


$$V(j\omega) = H(j\omega)W(j\omega) = X(j\omega) \text{ if } \omega_b < \omega_{co} < 2\omega_c - \omega_b$$

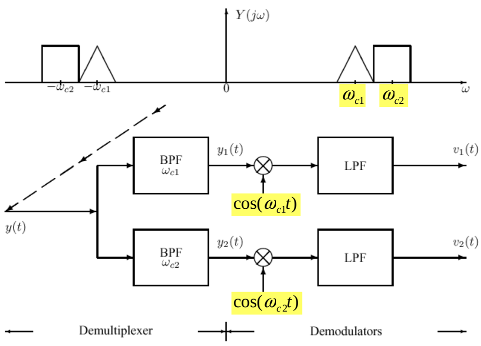
Frequency-Division Multiplexing (FDM)

- Shifting spectrum of signal to higher frequency:
  - Permits transmission of low-frequency signals with high-frequency EM waves
  - By allocating a frequency band to each signal multiple bandlimited signals can share the same channel
  - AM radio: 530-1620 kHz (10 kHz bands)
  - FM radio: 88.1-107.9 MHz (200 kHz bands)

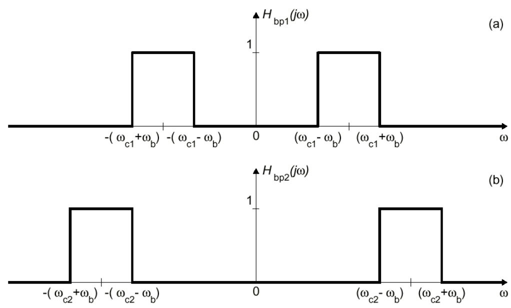
FDM Block Diagram (Xmitter)



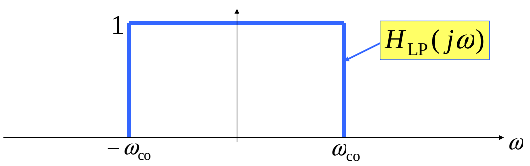
Frequency-Division De-Mux



Bandpass Filters for De-Mux



Pop Quiz: FT thru LPF



Input $x(t) \quad \leftrightarrow \quad X(j\omega) = \sum_{k=-\infty}^{\infty} 4\pi\delta(\omega - 30\pi k)$

If the output is $y(t) = 2$, then find a value for ω_{co}

Sampling and Reconstruction (Fourier View)

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- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- Review of FT properties
  - Convolution ←→ multiplication
  - Frequency shifting
  - Review of AM

Table of FT Properties

$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$

Delay Property

$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$

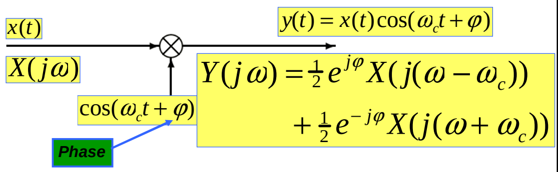
Frequency Shifting

$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$

Scaling

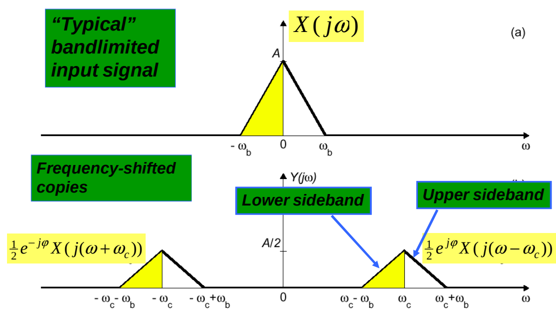
$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$

Amplitude Modulator

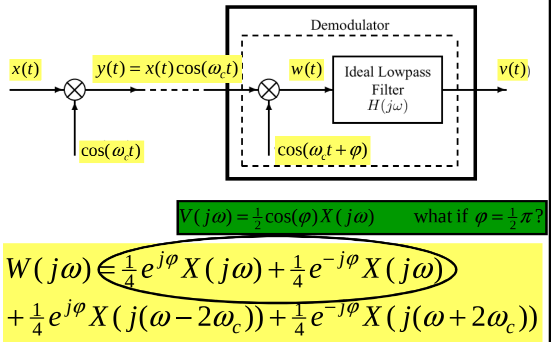


- *x(t)* modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of *X(jω)*.

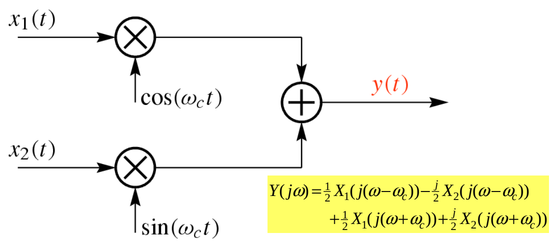
DSBAM: Frequency-Domain



DSBAM Demod Phase Synch

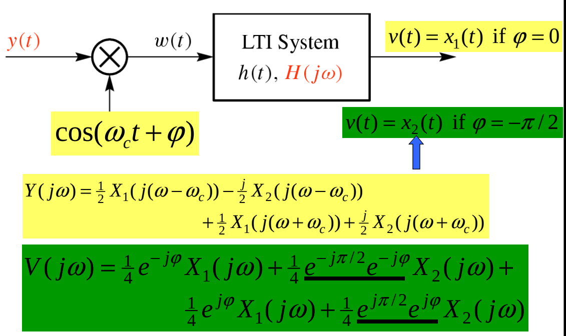


Quadrature Modulator

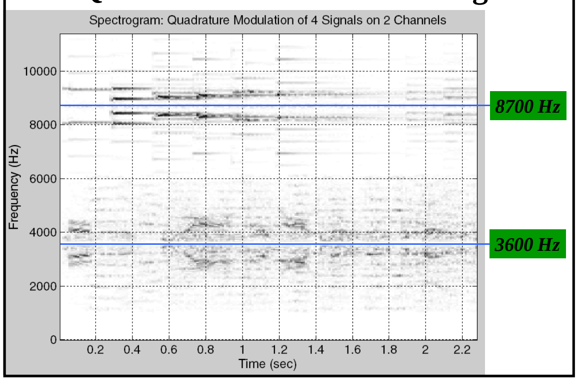


TWO signals on ONE channel: "out of phase"
Can you "separate" them in the demodulator ?

Demod: Quadrature System

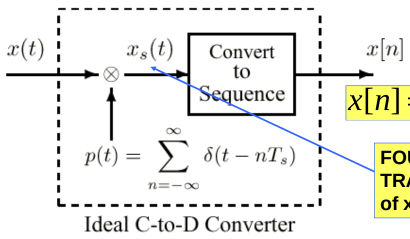


Quadrature Modulation: 4 sigs



Ideal C-to-D Converter

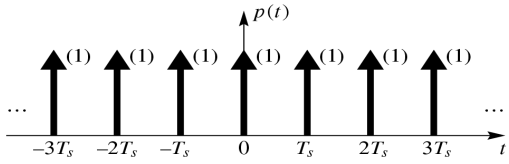
- **Mathematical Model for A-to-D**



$x[n] = x(nT_s)$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

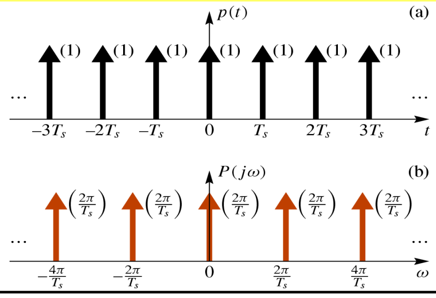
$$\omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

Fourier Series

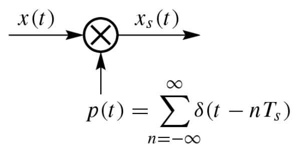
FT of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \leftrightarrow \quad P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$



$$\omega_s = \frac{2\pi}{T_s}$$

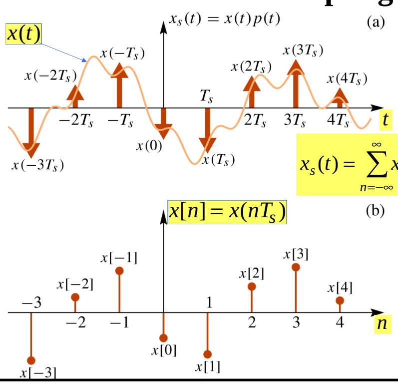
Impulse Train Sampling



$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

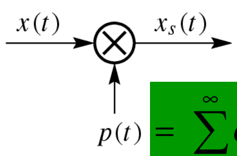
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Illustration of Sampling



$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Sampling: Freq. Domain



$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

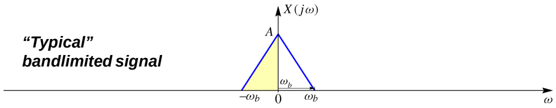
$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\frac{1}{T_s}e^{jk\omega_s t} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

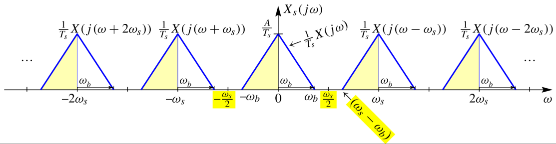
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

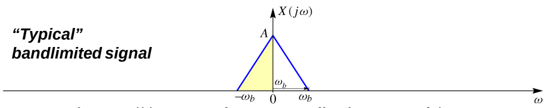


$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

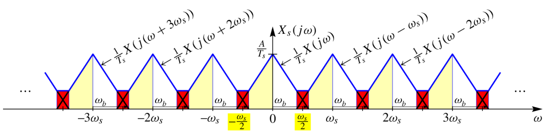


Aliasing Distortion

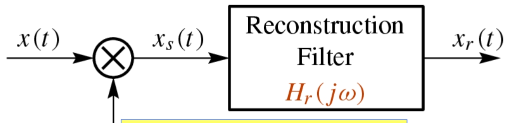
"Typical" bandlimited signal



we have aliasing distortion.



Reconstruction of *x*(*t*)

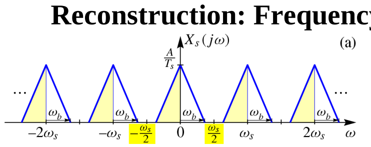


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

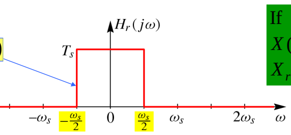
$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

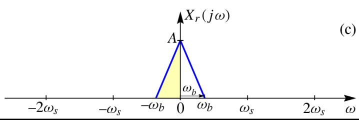
Reconstruction: Frequency-Domain



If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

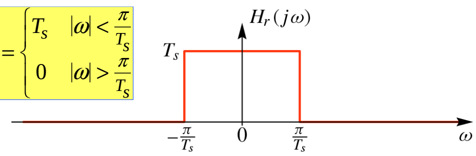
$H_r(j\omega)$



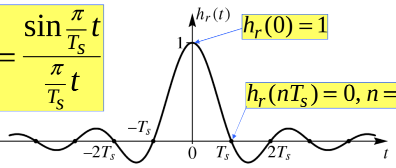


Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



$h_r(0) = 1$

$h_r(nT_s) = 0,\ n = \pm1, \pm2, \ldots$

Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

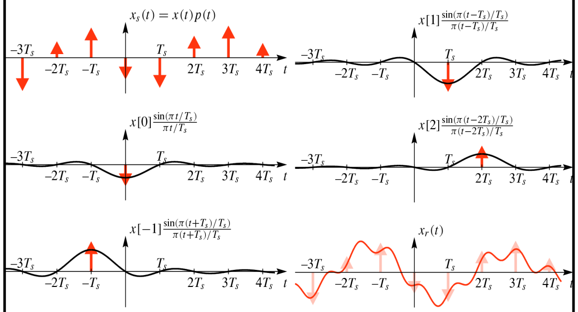
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

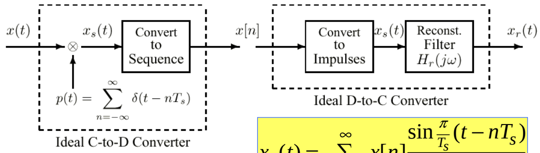
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$