

Advanced Digital Signal Processing

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Introduction to the Fourier Transform

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- Review
 - Frequency Response
 - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Relation to Fourier Series

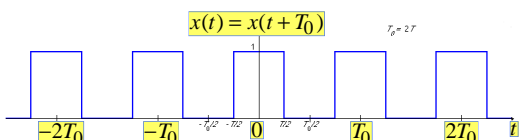
- Examples of Fourier transform pairs

Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
 - ????????????
- Finite Length
- Not Periodic
- Limit of Square Wave as Period \rightarrow infinity
 - Intuitive Argument



Fourier Series: Periodic $x(t)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

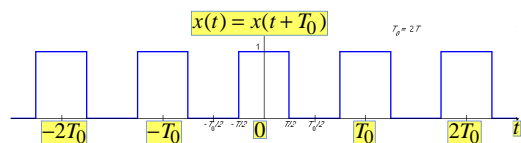
$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

Square Wave Signal



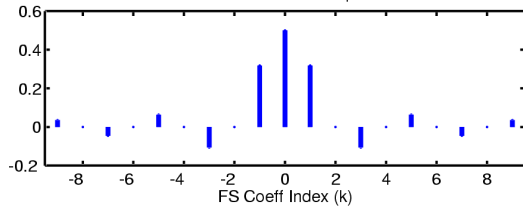
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k T_0} \Big|_{-T_0/4}^{T_0/4}}{-j\omega_0 k T_0} = \frac{e^{-j\pi k / 2} - e^{j\pi k / 2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

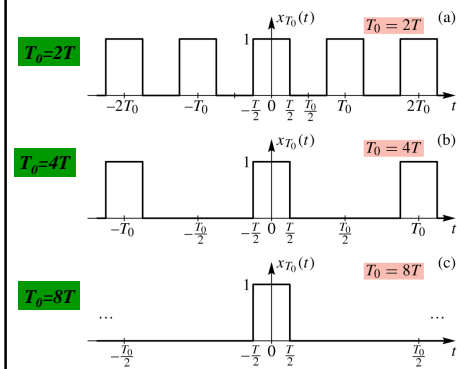
Fourier Series Coeffs for Square Wave



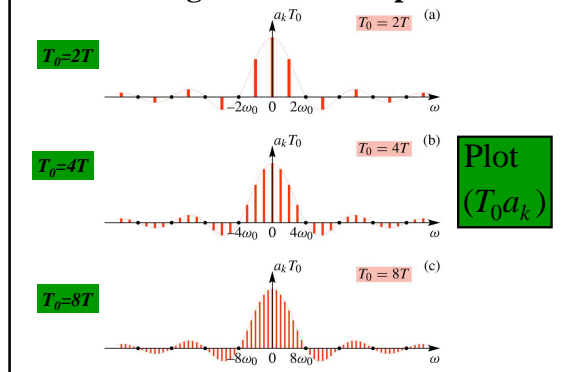
What if $x(t)$ is not periodic?

- Sum of Sinusoids?
 - Non-harmonically related sinusoids
 - Would not be periodic, but would probably be non-zero for all t .
- Fourier transform
 - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
 - can represent signals that are identically zero for negative t . !!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_k t} \left(\frac{2\pi}{T_0} \right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} = d\omega \quad \lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} k = \omega \quad \lim_{T_0 \rightarrow \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Fourier Synthesis}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Fourier Analysis}$$

Example 1: $x(t) = e^{-at} u(t)$

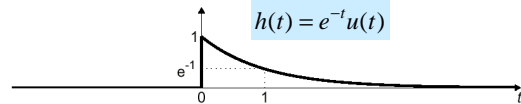
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a+j\omega} \Big|_0^{\infty} = \frac{1}{a+j\omega} \quad a > 0$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

Frequency Response

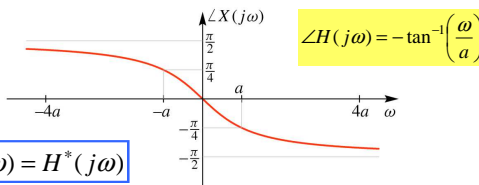
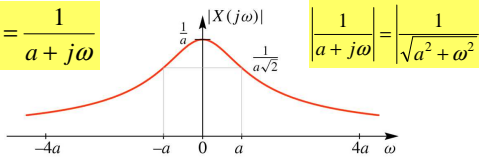
- Fourier Transform of $h(t)$ is the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a+j\omega}$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

Example 2:

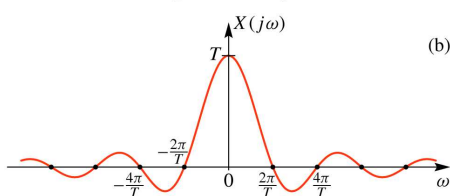
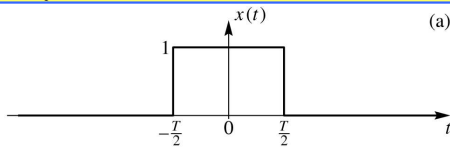
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$



Example 3:

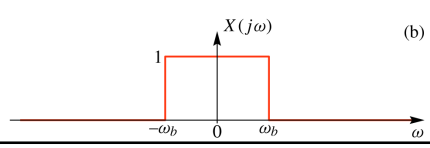
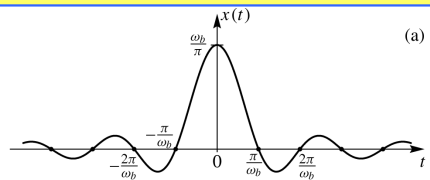
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} 1 e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega_b t}}{jt} \Big|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi} \frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$



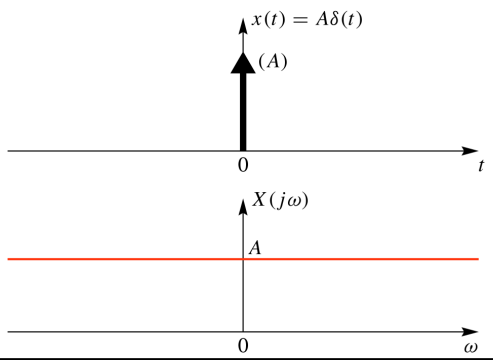
Example 4: $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

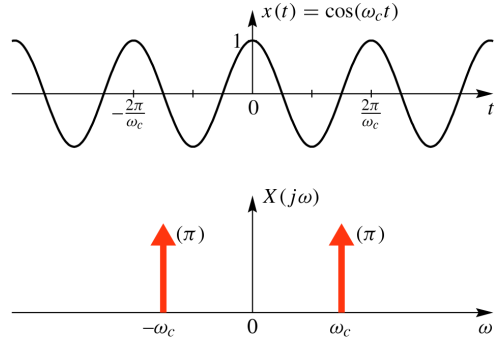


Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform Properties

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- The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
 - [Convolution](#) property
 - [Multiplication](#) property

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

Time - Domain \Leftrightarrow Frequency - Domain

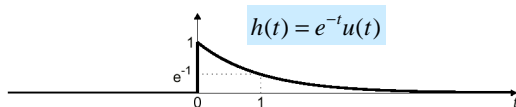
$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- Manipulate the [“Frequency Spectrum”](#)
- Analog Communication Systems
 - AM: Amplitude Modulation; FM
 - What are the [“Building Blocks”](#) ?
 - [Abstract Layer](#) not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
 - aka Modulators, Mixers or Multipliers: $x(t)p(t)$

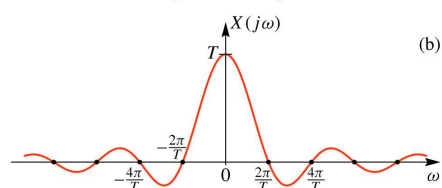
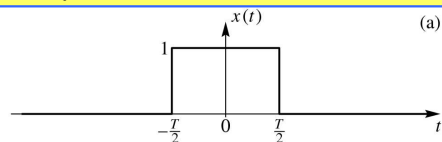
Frequency Response

- Fourier Transform of $h(t)$ **is** the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



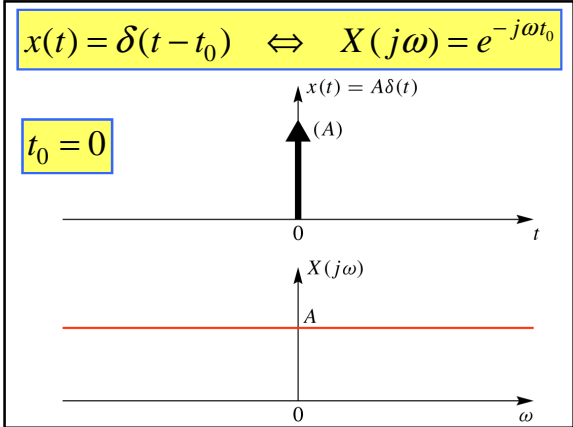
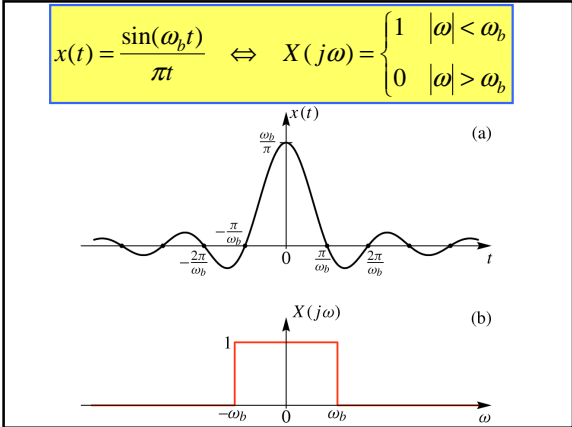


Table of Fourier Transforms

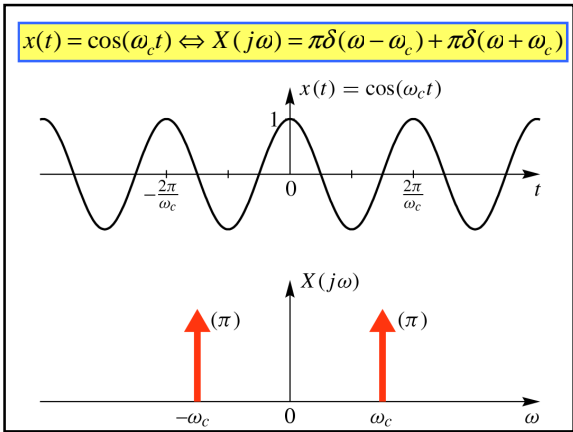
$x(t) = e^{-t}u(t) \Leftrightarrow X(j\omega) = \frac{1}{1 + j\omega}$

$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$

$x(t) = \frac{\sin(\omega_b t)}{\pi} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$

$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$

$x(t) = e^{j\omega_c t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_c)$



Fourier Transform of a General Periodic Signal

- If $x(t)$ is periodic with period T_0 ,

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$

Square Wave Signal

$x(t) = x(t + T_0)$

Plot of a square wave $x(t)$ versus t . The signal is periodic with period T_0 . It has a value of 1 from $t = 0$ to $t = T_0/2$ and a value of -1 from $t = T_0/2$ to $t = T_0$. The x-axis is marked at $-2T_0$, $-T_0$, $-T_0/2$, 0 , $T_0/2$, T_0 , and $2T_0$.

$a_k = \frac{1}{T_0} \int_0^{T_0/2} (1) e^{-j\omega_0 k t} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 k t} dt$

$a_k = \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Big|_0^{T_0/2} - \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Big|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$

Square Wave Fourier Transform

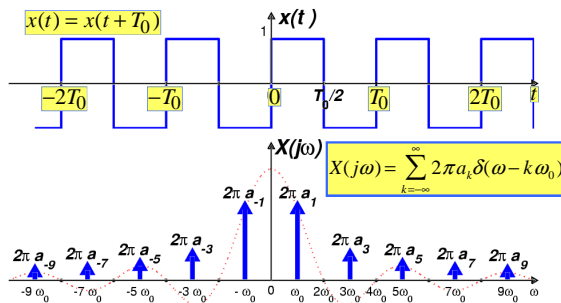


Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

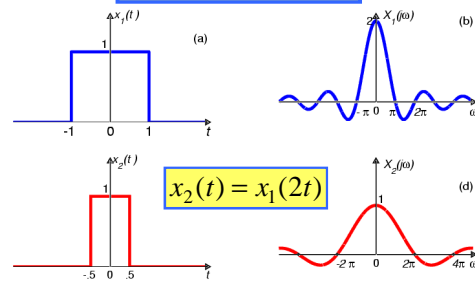
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

$x(2t)$ shrinks; $\frac{1}{2} X(j\frac{\omega}{2})$ expands

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$



Uncertainty Principle

- Try to make $x(t)$ shorter
 - Then $X(j\omega)$ will get wider
 - Narrow pulses have wide bandwidth
- Try to make $X(j\omega)$ narrower
 - Then $x(t)$ will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

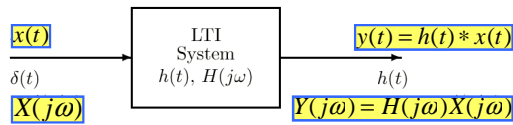
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$Y(j\omega) = H(j\omega)X(j\omega)$$

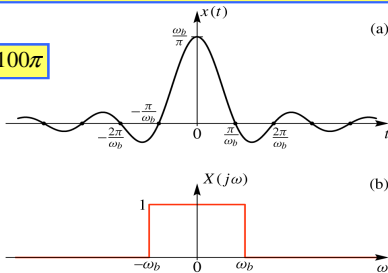
Convolution Example

- Bandlimited **Input** Signal
– “sinc” function
- Ideal LPF (Lowpass Filter)
– $h(t)$ is a “sinc”
- **Output** is Bandlimited
– Convolve “sincs”

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

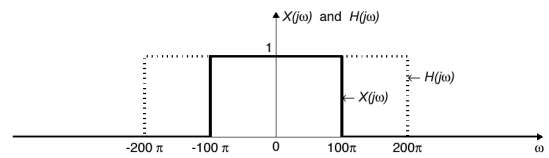
$$\omega_b = 100\pi$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

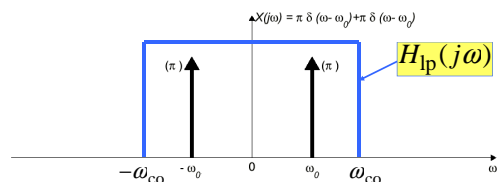
$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

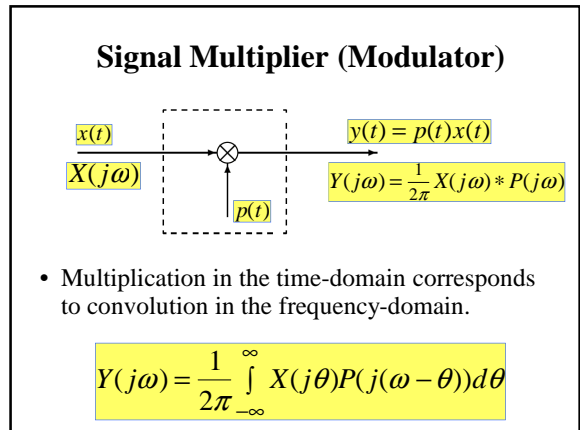
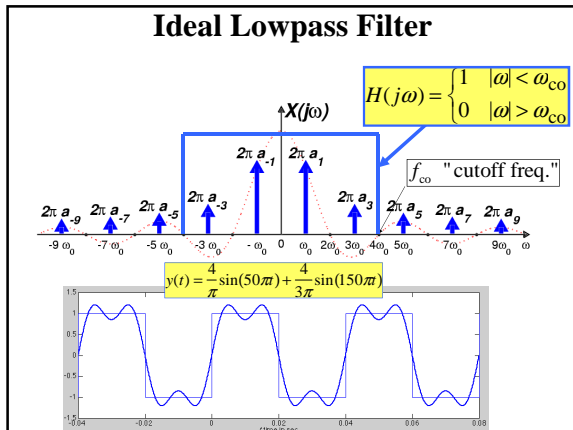
$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] \\ &= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0) \\ y(t) &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter



$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

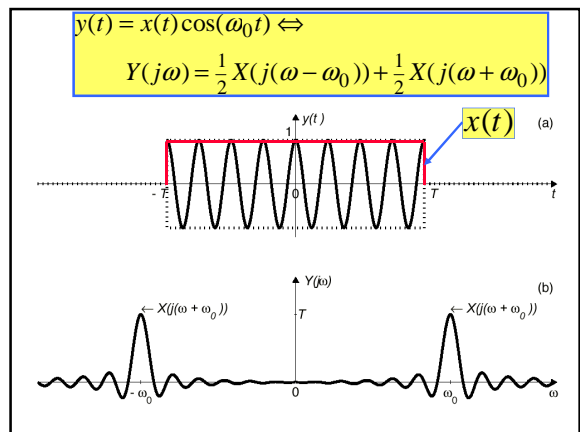


Frequency Shifting Property

$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt = X(j(\omega - \omega_0))$$

$y(t) = \frac{\sin 7t}{\pi} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & \text{elsewhere} \end{cases}$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right)$$

Multiply by $j\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} (e^{-at} u(t)) = -ae^{-at} u(t) + e^{-at} \delta(t)$$

$$= \delta(t) - ae^{-at} u(t)$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

$y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$p(t) = \cos(\omega_0 t) \Leftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$y(t) = x(t) \cos(\omega_0 t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

$$\int_{-\infty}^{\infty} x(t-t_d)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_d)} d\tau = e^{-j\omega t_d} X(j\omega)$$

For example, $e^{-a(t-5)}u(t-5) \Leftrightarrow \frac{e^{-j\omega 5}}{a+j\omega}$

Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of “theorems” or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution.*

FT of Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 t} dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\therefore P(j\omega) = \left(\frac{2\pi}{T_0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Convolution Example 2

