Advanced Digital Signal Processing

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1

Fourier Series Coefficients

2

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: **$x(t+T_0) = x(t)$**
  - Spectrum from the Fourier Series

3

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era

- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

4

Joseph Fourier

lived from 1768 to 1830

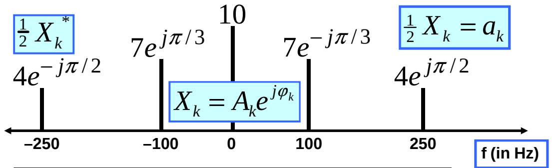
Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

5

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N}\left\{a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t}\right\}$$

6

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

7

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

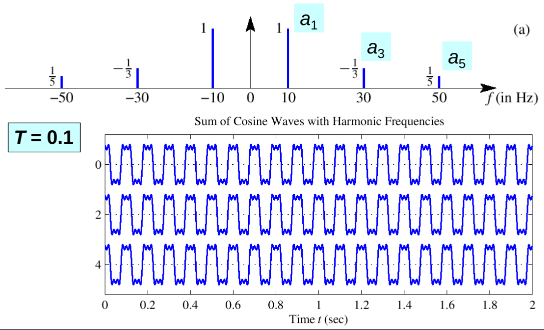
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

8

Harmonic Signal (3 Freqs)



9

SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

10

STRATEGY: *x*(*t*) → a_k

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

11

INTEGRAL Property of *exp*(*j*)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

12

ORTHOGONALITY of *exp(j)*

- PRODUCT of *exp(+j)* and *exp(-j)*

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}\,dt$$

13

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \tfrac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}dt$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \sum_{k=-\infty}^{\infty} a_k\left(\tfrac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}dt\right) = a_\ell$$

Integral is zero except for $k = \ell$

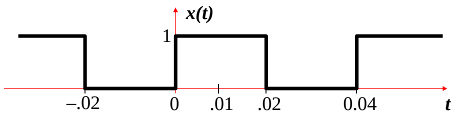
$$\Rightarrow a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

14

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



15

FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \tfrac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

16

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

17

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

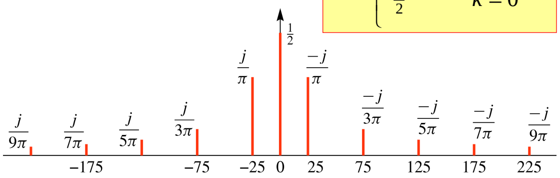
$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

18

Spectrum from Fourier Series

$\omega_0 = 2\pi/(0.04) = 2\pi(25)$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm1, \pm3, \dots \\ 0 & k = \pm2, \pm4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

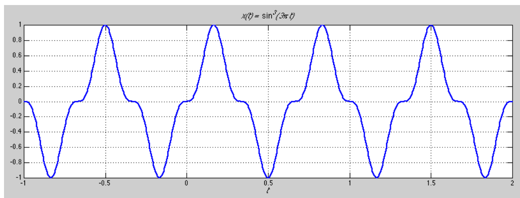
$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t) dt \qquad \text{(DC component)}$$

Fourier Series & Spectrum

Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

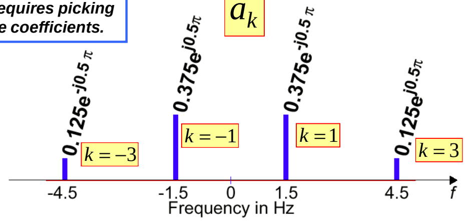
Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.

a_k



STRATEGY: $x(t) \rightarrow a_k$

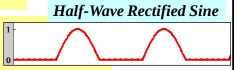
- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\omega_0 kt} dt$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \tfrac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\, e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

25

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

$$= \tfrac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \tfrac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\tfrac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \pm\frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$

26

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

27

Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

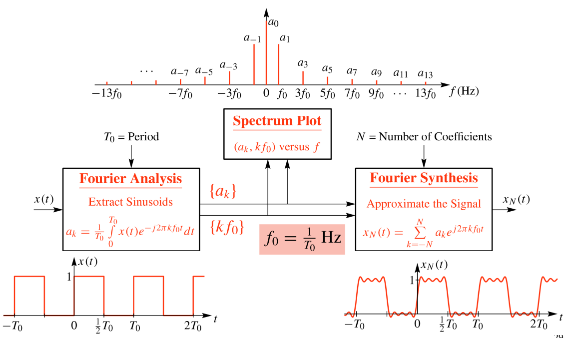
- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

$a_{-k} = a_k^*$ when $x(t)$ is real

28

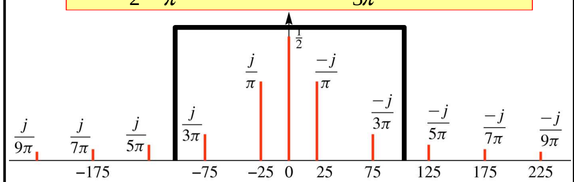
Fourier Series Synthesis

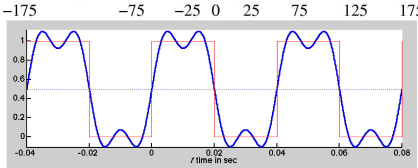


29

Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

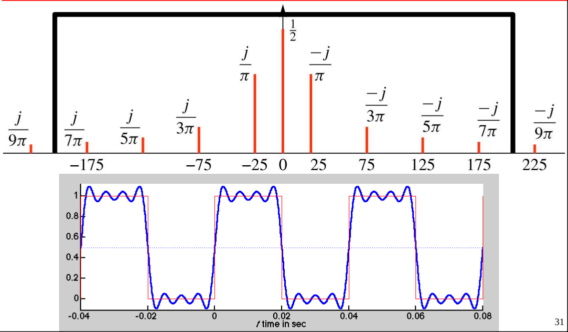




30

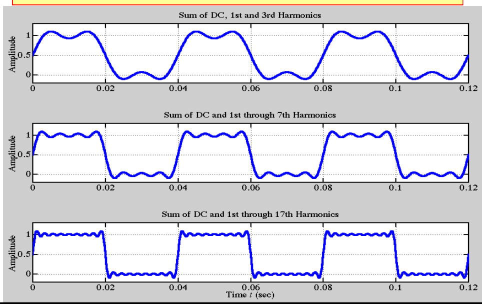
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$



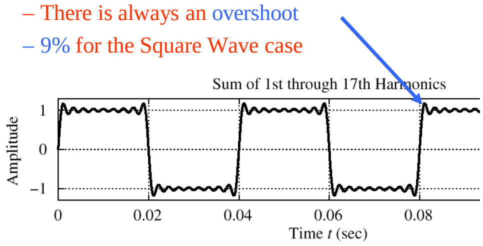
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of $x(t)$
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Series Demos

- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
  - http://users.ece.gatech.edu/mcclella/2025/Fsdemo_Slabaugh/fourier.html
- MATLAB GUI: fseriesdemo
  - http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html