Advanced Digital Signal Processing

Prof. Nizamettin AYDIN

naydin@yildiz.edu.tr
naydin@ieee.org
http://www.yildiz.edu.tr/~naydin

1

Digital Signal Processing

Some basics:

Sinusoids

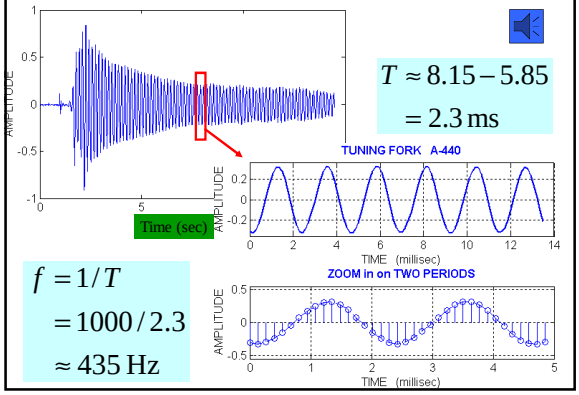
2

What's a signal

- A signal can be defined as
  - a pattern of variations of a physical quantity that can be manipulated, stored, or transmitted by physical process.
  - an information variable represented by physical quantity.
    - For digital systems, the variable takes on discrete values.
- In the mathematical sense it is a function of time, x(t), that carries an information.

3

TUNING FORK A-440 Waveform



$T \approx 8.15 - 5.85$
$= 2.3\,\text{ms}$

$f = 1/T$
$= 1000/2.3$
$\approx 435\,\text{Hz}$

4

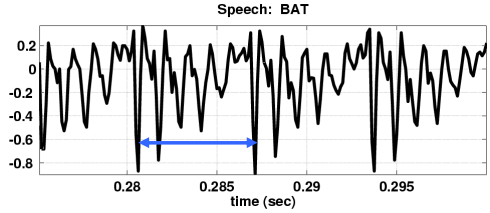
SPEECH EXAMPLE

- More complicated signal (BAT.WAV)
- Waveform ***x(t)*** is NOT a Sinusoid
- Theory will tell us
  - ***x(t)*** is approximately a sum of sinusoids
  - FOURIER ANALYSIS
    - Break ***x(t)*** into its sinusoidal components
  - Called the FREQUENCY SPECTRUM

5

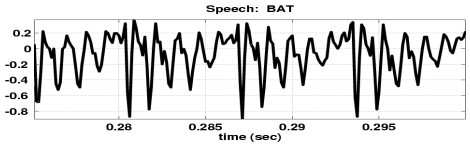
Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



6

One-dimensional continuous-time signal



- This speech signal is an example of one-dimensional continuous-time signal.
- Can be represented mathematically as a function of single independent variable (t).

7

Two-dimensional stationary signal

- This is a two dimensional signal (an image)
- A spatial pattern not varying in time
- Represented mathematically as a function of two spatial variables (x,y)
- However, videos are time-varying images that involves three independent variables (x,y,t)

8

DIGITIZE the WAVEFORM

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- Sample at 11,025 samples per second
  - Called the SAMPLING RATE of the A/D
  - Time between samples is
    - 1/11025 = 90.7 microsec
- Output via D/A hardware (at F_{samp})

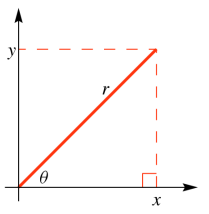
9

STORING DIGITAL SOUND

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
- 16-bit samples
- Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 x (16/8) x 60 x 44100 = 10.584 Mbytes

10

SINE and COSINE functions



$$\sin\theta = \frac{y}{r} \implies y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \implies x = r\cos\theta$$

McClellan, Schafer and Yoder, *Signal Processing First*, ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

11

SINES and COSINES

- Always use the COSINE FORM

$$A\cos(2\pi(440)t+\varphi)$$

- Sine is a special case:

$$\sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$$

12

SINUSOIDAL SIGNAL

$$A\cos(\omega t+\varphi)$$

- **FREQUENCY** ω
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega=(2\pi)f$$

- **PERIOD** (in sec)

$$T=\frac{1}{f}=\frac{2\pi}{\omega}$$

- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

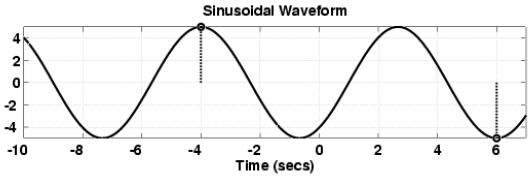
13

EXAMPLE of SINUSOID

- Given the Formula

$$5\cos(0.3\pi t+1.2\pi)$$

- Make a plot



PLOT COSINE SIGNAL

$$5\cos(0.3\pi t+1.2\pi)$$

- Formula defines A, ω, and ϕ

$$A=5$$
$$\omega=0.3\pi$$
$$\varphi=1.2\pi$$

15

PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(0.3\pi t+1.2\pi)$$

- Determine **period**:

$$T=2\pi/\omega=2\pi/0.3\pi=20/3$$

- Determine a **peak** location by solving

$$(\omega t+\varphi)=0 \quad \Rightarrow (0.3\pi t+1.2\pi)=0$$

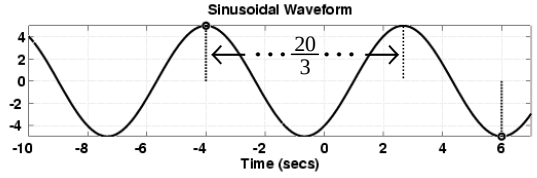
- **Zero** crossing is T/4 before or after
- Positive & Negative peaks spaced by T/2

16

PLOT the SINUSOID

$$5\cos(0.3\pi t+1.2\pi)$$

- Use T=20/3 and the peak location at t=-4



TIME-SHIFT

- In a mathematical formula we can replace t with t-t_m

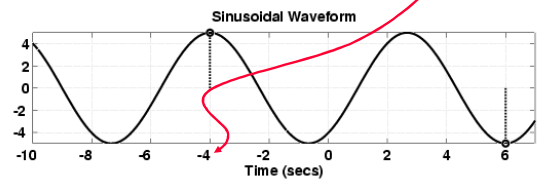
$$x(t-t_m)=A\cos(\omega(t-t_m))$$

- Then the t=0 point moves to t=t_m
- Peak value of $\cos(\omega(t-t_m))$ is now at t=t_m

18

TIME-SHIFTED SINUSOID

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



19

PHASE <--> TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t+\varphi)$$

- and we obtain: $-\omega t_m = \varphi$
- or, $t_m = -\dfrac{\varphi}{\omega}$

20

TIME-SHIFT

- Whenever a signal can be expressed in the form $x_1(t)=s(t-t_1)$, we say that $x_1(t)$ is time shifted version of $s(t)$
  - If t_1 is a + number, then the shift is to the right, and we say that the signal $s(t)$ has been *delayed* in time.
  - If t_1 is a - number, then the shift is to the left, and we say that the signal $s(t)$ was *advanced* in time.

21

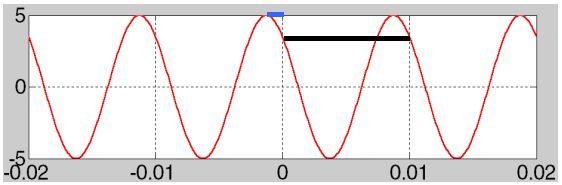
SINUSOID from a PLOT

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
  - Compute phase: $\phi = -\omega\, t_m$
- Measure height of positive peak: A

3 steps

22

(A, ω, ϕ) from a PLOT



$$T = \frac{0.01\text{sec}}{1\,\text{period}} = \frac{1}{100} \rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125\,\text{sec} \rightarrow \varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

23

PHASE is AMBIGUOUS

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t+\varphi+2\pi) = A\cos(\omega t+\varphi)$$

  - Thus adding any multiple of 2π leaves $x(t)$ unchanged

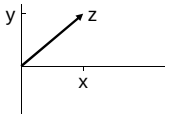
if $t_m = \dfrac{-\varphi}{\omega}$, then

$$t_{m_2} = \frac{-(\varphi+2\pi)}{\omega} = \frac{-\varphi}{\omega} - \frac{2\pi}{\omega} = t_m - T$$

24

COMPLEX NUMBERS

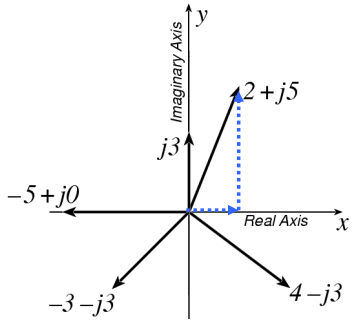
- To solve: $z^2 = -1$
  - z = ***j***
  - Math and Physics use z = ***i***
- Complex number: z = x + ***j*** y



Cartesian coordinate system

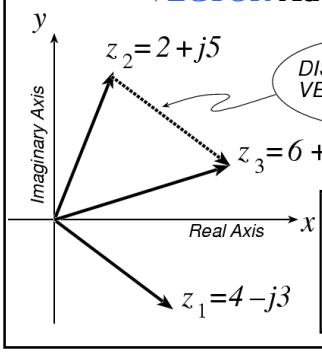
25

PLOT COMPLEX NUMBERS



26

COMPLEX ADDITION = VECTOR Addition

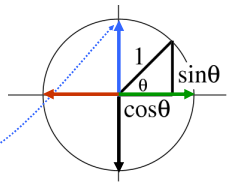


$$z_3 = z_1 + z_2$$
$$= (4 - j3) + (2 + j5)$$
$$= (4 + 2) + j(-3 + 5)$$
$$= 6 + j2$$

27

*** POLAR FORM ***

- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values



1 has angle of 0
j has angle of 0.5π
–1 has angle of π
– j has angle of 1.5π
also, angle of **–j could** be –0.5π = 1.5π –2π
because the PHASE is **AMBIGUOUS**

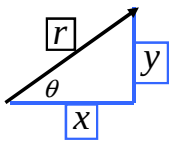
28

POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$
$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

Most calculators do Polar-Rectangular



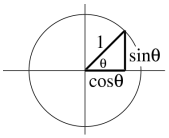
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Need a notation for POLAR FORM

29

Euler's FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

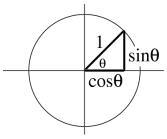
$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

30

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - **θ = ωt**
  - Angle changes vs. time
  - ex: ω=20π rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

31

cos = REAL PART

Real Part of Euler's $\cos(\omega t) = \Re e\{e^{j\omega t}\}$

General Sinusoid $x(t) = A\cos(\omega t + \varphi)$

So,

$$A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t+\varphi)}\}$$
$$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$

32

REAL PART EXAMPLE

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate: $x(t) = \Re e\left\{-3je^{j\omega t}\right\}$

Answer:

$$x(t) = \Re e\left\{(-3j)e^{j\omega t}\right\}$$
$$= \Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\} = 3\cos(\omega t - 0.5\pi)$$

33

COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

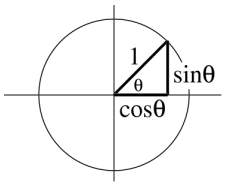
Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

34

TRIG FUNCTIONS

- Circular Functions



- Common Values
  - sin(*k*π) = 0
  - cos(0) = 1
  - cos(2*k*π) = 1 and cos((2*k*+1) π) = -1
  - cos((*k*+0.5) π) = 0

35

Basic properties of the sine and cosine functions

Property	Equation
Equivalence	$\sin\theta = \cos(\theta - \pi/2)$ or $\cos(\theta) = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + 2\pi k) = \cos\theta$, when k is an integer
Evenness of cosine	$\cos(-\theta) = \cos\theta$
Oddness of sine	$\sin(-\theta) = -\sin\theta$
Zeros of sine	$\sin(\pi k) = 0$, when k is an integer
Ones of cosine	$\cos(2\pi k) = 1$ when k is an integer.
Minus ones of cosine	$\cos[2\pi(k + \frac{1}{2})] = -1$, when k is an integer.

36

Some basic trigonometric identities

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2\sin\theta\cos\theta$
4	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

37

Sampling and plotting sinusoids

- Plot the following function

$$20\cos(2\pi(40)t - 0.4\pi)$$

- Must evaluate $x(t)$ at a discrete set of times, $t_n = nT_s$, where n is an integer

$$x(nT_s) = 20\cos(80\pi nT_s - 0.4\pi)$$

- T_s is called sample spacing or sampling period
- Matlab or Scilab program.
- Hypersignal

38

39

SPECTRUM Representation

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

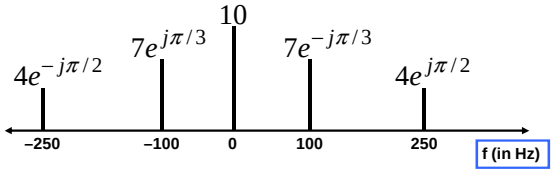
$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

40

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



41

Another FREQ. Diagram

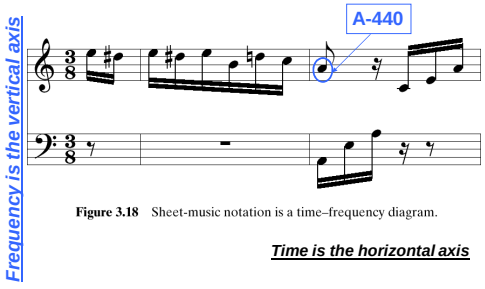


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

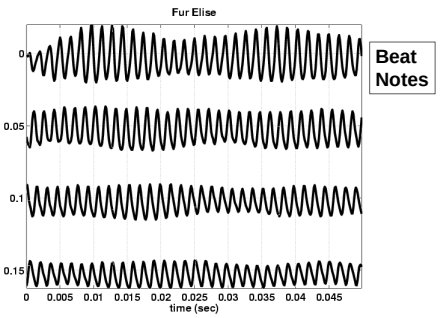
42

MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

43

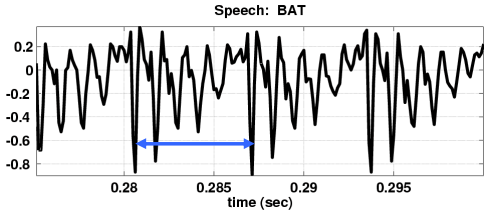
Fur Elise WAVEFORM



44

Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



45

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

46

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

47

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \tfrac{A}{2}e^{j7t} + \tfrac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

48

NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

49

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

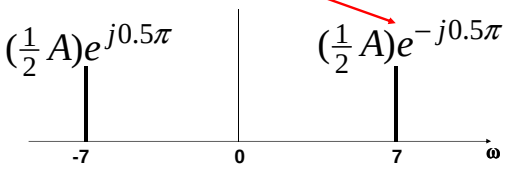
  - Positive freq. has phase = -0.5π
  - Negative freq. has phase = +0.5π

50

GRAPHICAL SPECTRUM

EXAMPLE of SINE

$$A\sin(7t) = \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

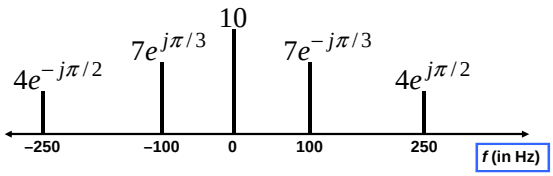


AMPLITUDE, PHASE & FREQUENCY are shown

51

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal *x(t)*?

52

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 -π/2
  - 7 +π/3
  - 10 **0**
  - 7 -π/3
  - 4 +π/2

Note the **conjugate phase**

DC is another name for zero-freq component
DC component always has **ϕ=0** or **π** (for real ***x(t)***)

53

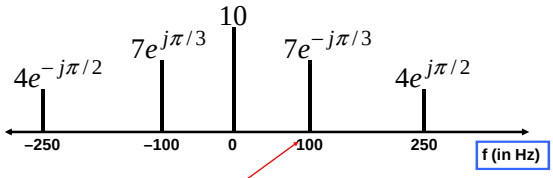
Add Spectrum Components-1

- **Frequencies:**
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- **Amplitude & Phase**
  - 4 -π/2
  - 7 +π/3
  - 10 0
  - 7 -π/3
  - 4 +π/2

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

54

Add Spectrum Components-2



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

55

Simplify Components

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

56

FINAL ANSWER

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

57

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\{X_k e^{j2\pi f_k t}\}$$

$$X_k = A_k e^{j\varphi_k}$$

Frequency = f_k

$$\Re e\{z\} = \tfrac{1}{2}z + \tfrac{1}{2}z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \{\tfrac{1}{2}X_k e^{j2\pi f_k t} + \tfrac{1}{2}X_k^* e^{-j2\pi f_k t}\}$$

58

Example: Synthetic Vowel

- Sum of 5 Frequency Components

f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	−0.185
1600	$(1657 - j13520)$	13,621	−1.449
1700	$4723 + j0$	4723	0

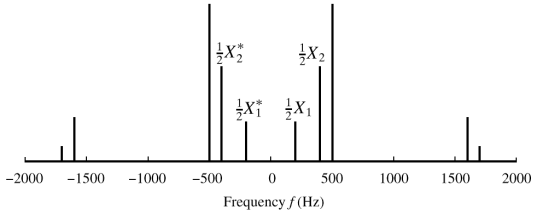
Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

59

SPECTRUM of VOWEL

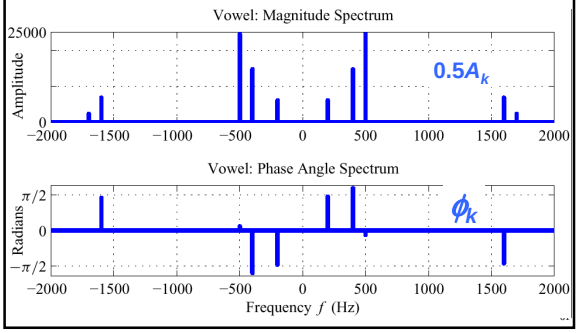
– Note: Spectrum has $0.5X_k$ (except X_{DC})

– Conjugates in negative frequency



60

SPECTRUM of VOWEL (Polar Format)



Vowel Waveform (sum of all 5 components)

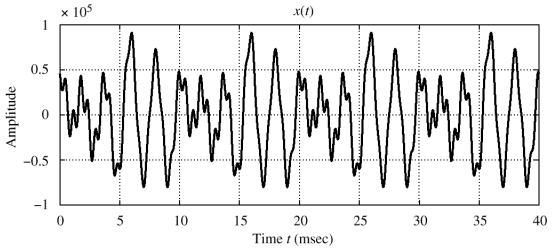


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

62

Periodic Signals, Harmonics & Time-Varying Sinusoids

63

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - Amp, Freq, Phase
- Recorded Signals
  - Speech
  - Music
  - No simple formula
- Plot & Sketches
  - $S(t)$ versus t
  - Spectrum
- MATLAB
  - Numerical
  - Computation
  - Plotting list of numbers

64

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

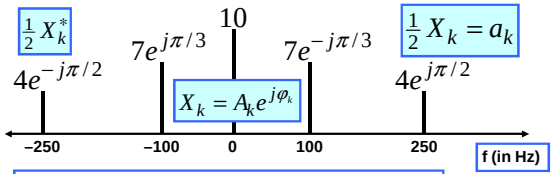
Chirps: $x(t) = \cos(\alpha t^2)$

Introduce Spectrogram Visualization **(specgram.m)**

65

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq

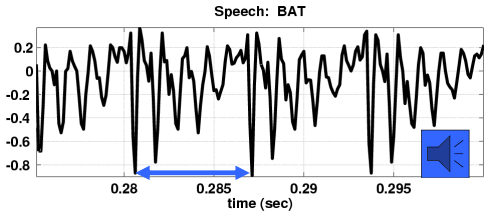


$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

66

SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



67

PERIODIC SIGNALS

- Repeat every T secs
  - Definition $x(t) = x(t+T)$
  - Example:

$$x(t) = \cos^2(3t)$$

$T = ?$

$T = \frac{2\pi}{3}$ $T = \frac{\pi}{3}$

  - Speech can be "quasi-periodic"

68

Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t) \; ?$$

Definition: Period is T

$$e^{j\omega(t+T)} = e^{j\omega t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega T} = 1 \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right) k = \omega_0 k$$

k = integer

69

Harmonic Signal Spectrum

Periodic signal can only have : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

70

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

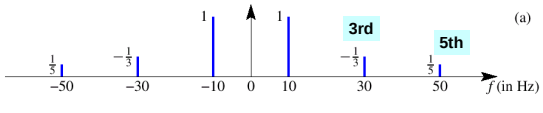
$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency (largest)

T_0 = fundamental Period (shortest)

71

Harmonic Signal (3 Freqs)

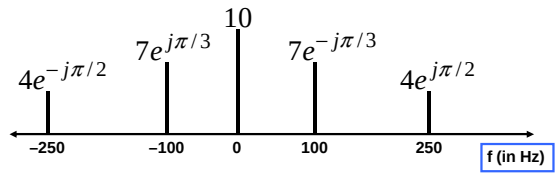


What is the fundamental frequency? **10 Hz**

72

Example

- Here's another spectrum:

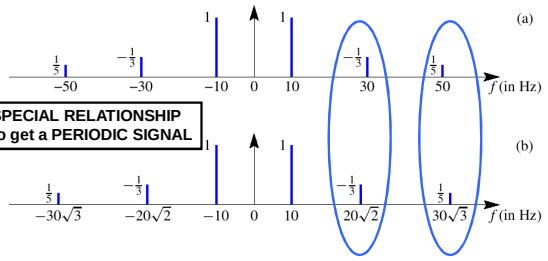


What is the fundamental frequency?

100 Hz ? **50 Hz ?**

73

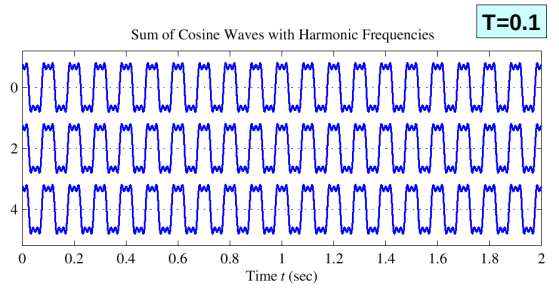
IRRATIONAL SPECTRUM



SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

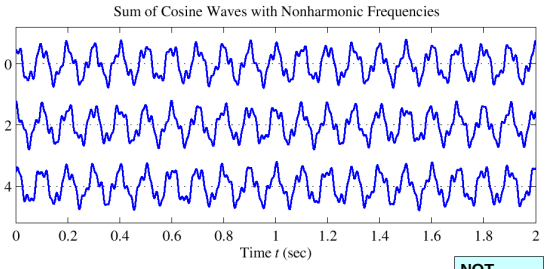
74

Harmonic Signal (3 Freqs)



75

NON-Harmonic Signal

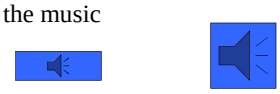


NOT PERIODIC

76

FREQUENCY ANALYSIS

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music



- Can a machine extract frequencies?
  - – Yes, if we COMPUTE the spectrum for $x(t)$
    - During short intervals

77

Time-Varying FREQUENCIES Diagram

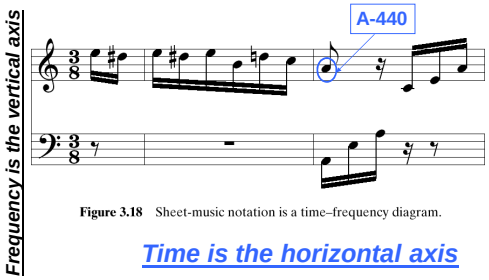


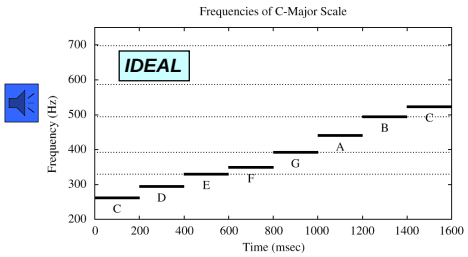
Figure 3.18 Sheet-music notation is a time–frequency diagram.

<u>Time is the horizontal axis</u>

78

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note



79

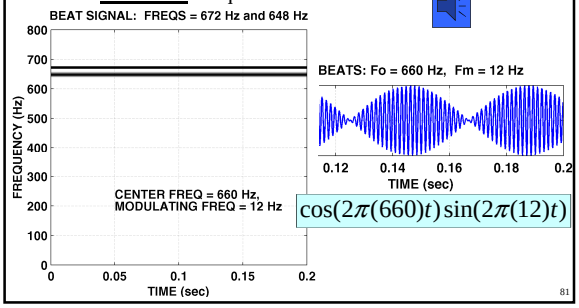
SPECTROGRAM

- SPECTROGRAM Tool
  - MATLAB function is **specgram.m**
  - SP-First has **plotspec.m** & **spectgr.m**
- **ANALYSIS** program
  - Takes x(t) as input &
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

80

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats



$\cos(2\pi(660)t)\sin(2\pi(12)t)$

81

AM Radio Signal

- Same as BEAT Notes

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

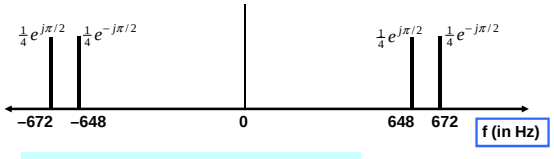
$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\frac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\frac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})$$

82

SPECTRUM of AM (Beat)

- 4 complex exponentials in AM:



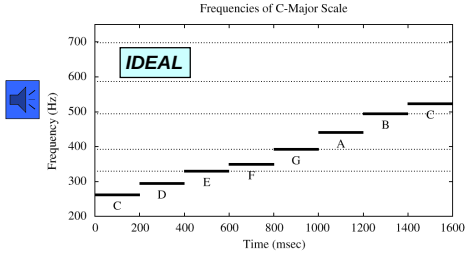
What is the fundamental frequency?

648 Hz ? **24 Hz ?**

83

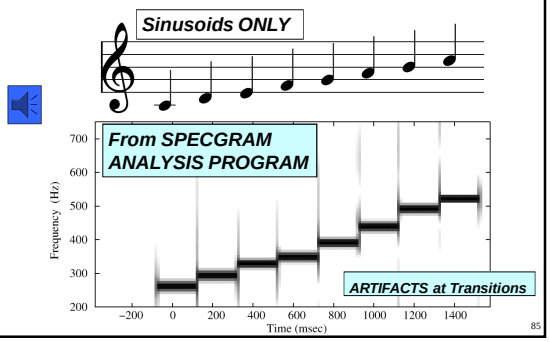
STEPPED FREQUENCIES

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note



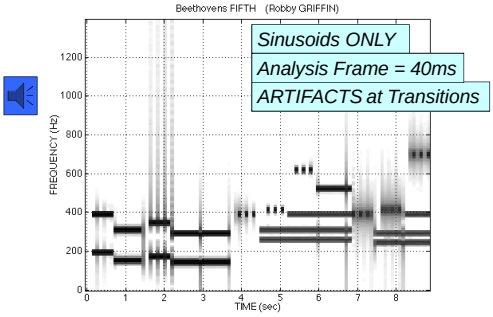
84

SPECTROGRAM of C-Scale



85

Spectrogram of LAB SONG



86

Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

87

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

88

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\pi f_0$$

Makes sense

89

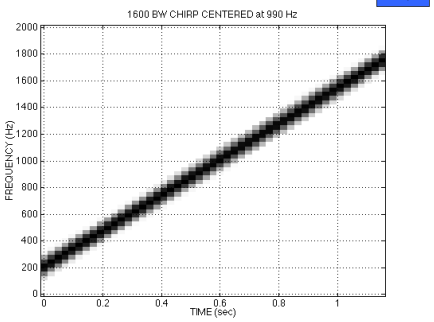
INSTANTANEOUS FREQ of the Chirp

- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

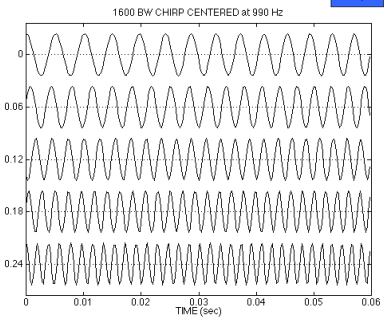
90

CHIRP SPECTROGRAM



91

CHIRP WAVEFORM



92

OTHER CHIRPS

ψ(t) can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

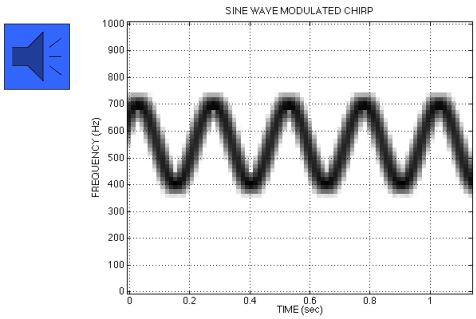
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

ψ(t) could be speech or music:

- FM radio broadcast

93

SINE-WAVE FREQUENCY MODULATION (FM)



94